THEORY OF
LIMIT DESIGN

BY J. A. VAN DEN BROEK

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THEORY OF LIMIT DESIGN

BY

J. A. VAN DEN BROEK, Ph.D.
Professor of Engineering Mechanics
University of Michigan

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PREFACE

Any theory is based on assumptions or premises. Any engineering theory involves "sense of value." For the last hundred and fifty years elastic stress analysis has been so intimately allied to theory of strength that the two have come to be regarded as almost synonymous.

Theory of strength involves two basic considerations, equilibrium and continuity. The former of these considerations is primary, the latter secondary. One of the significant contributions to theory of strength, if not in fact the most significant, is Euler's (1757) paper on the strength of columns (reference 10). This paper was published before stress analysis was born. It was pure limit design, as all earlier efforts had been up to that time.

For the purpose of supplementing the theory of equilibrium, two theories are at present offered, the theory of elasticity and the theory of limit design. The theory of elasticity presupposes elastic behavior and elastic stress distribution. It involves numerous assumptions which are little more than wishful thinking. To many it appears very inadequate. The theory of limit design presupposes ductile or semiductile stress distribution. In it, the emphasis is shifted from permissible safe stresses to permissible safe deformations. This theory is the subject of this book.

Acknowledgments

I am indebted to the *Michigan Technic* for Figs. 31 and 32; to *Civil Engineering* of the ASCE for Figs. 71, 72, and 73, and to the *Engineering Journal* of the Engineering Institute of Canada for numerous figures first published in this *Journal*. We received special co-operation from the Royal Aeronautical Society of London in the reproduction of Fig. 50, reference 1*m*. Figures 14 and 16 are taken from Professor E. O. Scott's thesis (reference 5); Figs. 7 and 33 to 38 are accredited to Dr. F. Panlilio's thesis (reference 8).

I am indebted to the late N. C. Kist, who made me ductility-conscious; and to the late C. M. Goodrich who made me limit-

design-conscious, although he had, himself, never referred to his design practices as "limit design." I am also indebted to colleagues and friends for constructive suggestions and criticisms, among whom I mention the following: Mr. Anders Bull, consulting engineer, Forest Hills, N. Y.; Dr. P. C. Hu, senior structural engineer, National Advisory Committee on Aeronautics; Mr. Armand Circé, formerly Dean École Polytechnique, Montreal; Dr. F. Panlilio, and Mr. Paul Coy, my colleagues at the University of Michigan, and Robert Fisher of Detroit and Arthur Witting of Ann Arbor.

J. A. VAN DEN BROEK

November 6, 1947

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Chapter 1

PHYSICAL PROPERTIES OF METALS

Stress we define as load intensity, load per unit area. In the case of a load uniformly distributed over an area, stress is

$$s = \frac{P}{A}$$

Strain we define as deformation intensity, deformation per unit of length. In the case of a deformation uniformly distributed along a length, strain is

$$e = \frac{\Delta l}{l}$$

Modulus of Elasticity we define as the ratio:

$$\frac{\text{Stress increment}}{\text{Strain increment}} : E = \frac{\Delta s}{\Delta e}$$

Elasticity. We define an elastic material as a material possessed of a constant modulus of elasticity. Elasticity is frequently defined as the property of a material by means of which it returns to its original shape when the cause of the deformation is completely removed. The first definition implies the second, but the second definition does not imply the first.

Stress–Strain Curves. A stress–strain curve is a graphical representation of the stress–strain relationship of a material. Conventionally, stress is measured along the ordinate, and strain along the abscissa. In a simple tension test the cross-section dimensions decrease as the axial dimensions increase. Thus the cross-section area of the specimen constantly decreases as the test progresses. The true stress, if uniform stress distribution is assumed, would be the load divided by the area prevailing at the instant the load is measured. Most commonly the stress represented on stress–strain curves is the load divided by the original cross-section area. The correction due to the varying cross-section area is commonly

ignored. In the various stress–strain curves shown in this book we have followed this common practice. These curves should thus be considered as first approximations to true stress–strain curves. True tensile stress–strain curves would run slightly higher, and true compressive stress–strain curves slightly lower, than the ones shown.

In terms of stress–strain curves, a material is *elastic* when the stress–strain curve is a straight line through the origin for either increasing or decreasing loads.

The **Elastic-Limit** stress, in the light of our definition, is the same as the proportional limit stress; it is the stress at which the stress–strain curve ceases to be a straight line.

The **Yield Stress** is the stress at which a pronounced deformation takes place under constant load, or under a slightly increasing load. In mild steel the yield stress is substantially equivalent to the elastic-limit stress. The matter, however, is complicated in that we may have an upper and a lower yield stress. This upper yield stress appears extremely sensitive to very minor errors in testing, especially alignment, to residual stresses, and to rate of loading. The upper yield stress of mild steel may be a bothersome factor in refined research. However, it seems to require but small consideration in theory of strength.

Metals such as aluminum and magnesium do not exhibit the pronounced and sudden yielding phenomenon as does mild steel. The yield stress is thus arbitrarily defined as the stress corresponding to a deformation 0.002 in./in. greater than the deformation that would have existed if the material had behaved perfectly elastically. On the stress–strain curve (see Fig. 38) the yield stress is found by projecting a line parallel to the slope of the initial stress–strain curve, from a point on the abscissa representing 0.002 strain, until it intersects the stress–strain curve.

Ductility. Ductility is the property of a material by virtue of which it deforms extensively under a constant or slightly increasing stress. Ductility is most ideally exemplified by mild steel. Figure 1 is synthesized from Figs. 2, 3, and 4. The portion AB of Fig. 1 represents purely elastic behavior of mild steel. When a stress corresponding to A' (from 34,000 to 40,000 psi in the case of mild steel) is reached, the material suddenly yields locally an amount represented by BD, Fig. 1. The distance BD, which represents yielding under a constant stress in the case of mild steel, is 10 to 20 times as large as the purely elastic deformation

represented by $A'B$. In an ordinary tension or compression test this yielding appears to be gradual instead of sudden; that is, a number of readings may be taken, and several experimental points may be plotted (see curves 10 and 18 on Fig. 5). This fact must be attributed to experimental error. We cannot measure strain at a point. We measure the deformation over a finite length, divide that deformation by the length and obtain a value of average deformation. If that deformation is uniformly distributed, it represents strain at any point within the gage length. In the case of mild steel the first ductile yielding is not uniformly distributed, but is extremely localized. Not only does it manifest itself by the strain-gage reading, but also, if the specimen be polished, the failing of the crystal grain is manifested by the appearance of fine lines (Lüder lines) on the surface of the specimen. If the specimen is coated with mill scale, it manifests itself by the cracking of the mill scale. This cracking of the mill scale may be detected by ear. If testing machines were loading machines, the deformation BD (Fig. 1) would take place suddenly. Testing machines generally are straining machines, and thus, if the gage length is 8 in., the machine may be stopped and a reading taken halfway between B and D. Such a reading would mean that, of the material between the gage points, a portion of 4 in. had exceeded the elastic limit while the remaining 4 in. still remained completely unchanged. After a strain corresponding to point D in Fig. 1 is reached, progressive straining takes place under slowly increasing stresses. The maximum strain is represented by point G, Fig. 1. The significant fact to realize is that the distance $A''G$ is from 200 to 300 times as large as the distance $A'B$.

Figure 5 shows stress-strain curves of various aluminum, magnesium and steel alloy sheet metals tested transverse to the direction of rolling. Figure 6 shows similar records of sheet metal tested in direction of rolling.

On Fig. 7 is shown a picture of a hot-rolled square steel bar, bent cold upon itself. It appears obvious from the figure that in the bend the steel fibers on the outside are excessively elongated, and on the inside excessively shortened. The density of the steel remaining substantially constant, on the outside of the bend the bar has become narrower, while on the inside it has become wider. This property of ductility, excessive deformation under substantially constant stress, permits the cold drawing of steel into thin

wires, and stamping, bending, and cold forming. It is generally recognized as a highly desirable property. In the evaluation of

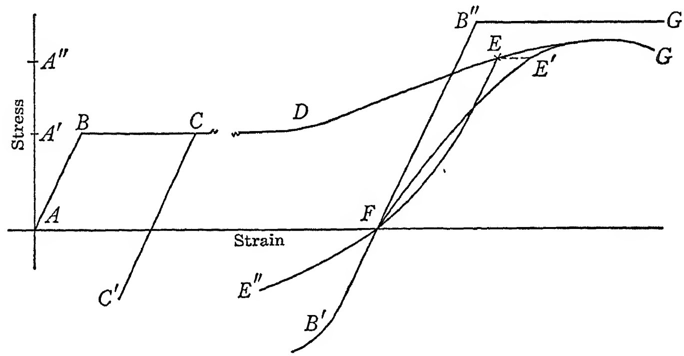


Fig. 1. Stress–Strain Properties of Cold-Worked Mild Steel. A composite diagram synthesized from Figs. 2 and 3.

such materials as magnesium or aluminum for structural purposes, it is frequently the determining factor. In view of the great importance that this property of ductility unquestionably holds, it

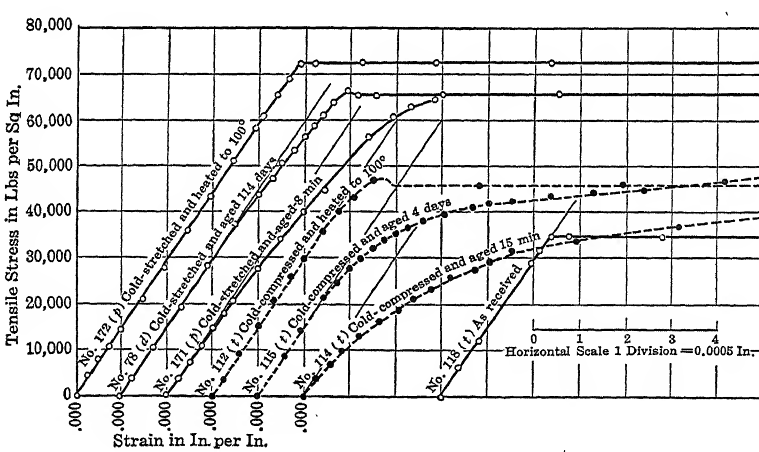


Fig. 2. Stress–Strain Curves of Cold-Worked Mild Steel Tested in Tension +. Solid lines mean as received or cold-worked in tension +. Dash lines mean cold-worked in compression −.

is strange that up to the present time it has received but scant attention in any formal logic directed to the study of strength of structures.

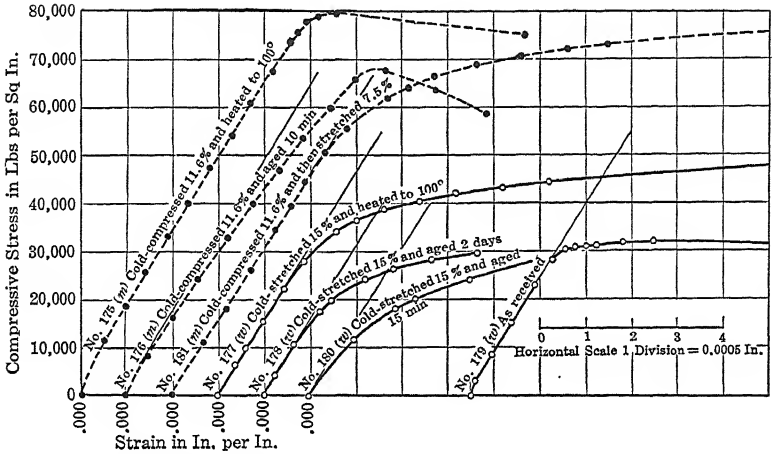


FIG. 3. Stress–Strain Curves of Cold-Worked Mild Steel Tested in compression −. Solid lines mean as received or cold-worked in tension +. Dash lines mean cold-worked in compression −.

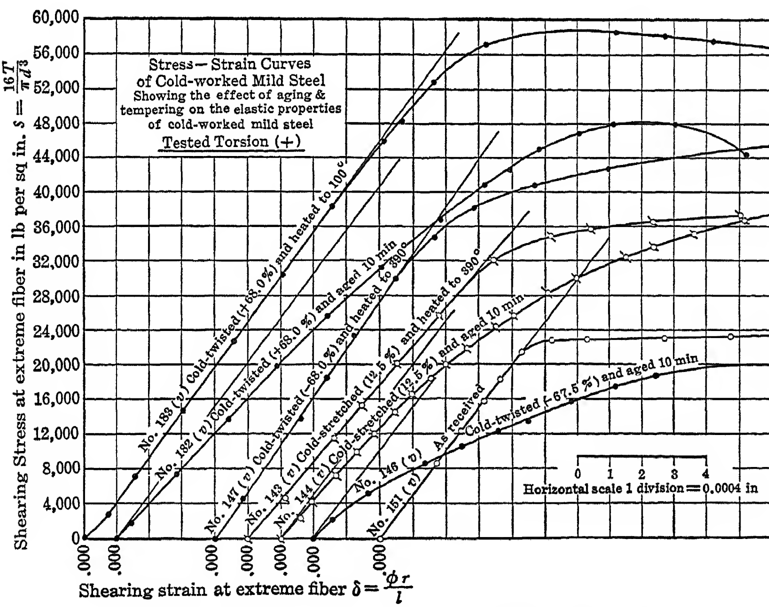


FIG. 4. Stress–Strain Curves of Cold-Worked Mild Steel Tested in Torsion +. Specimens were cold-worked both + and −.

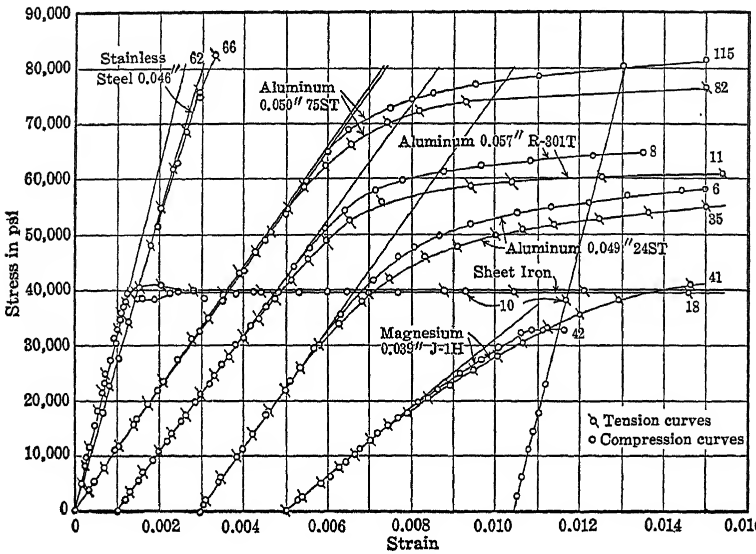


FIG. 5. Stress—Strain Curves of Sheet Metals Tested *Transverse* to Direction of Rolling.

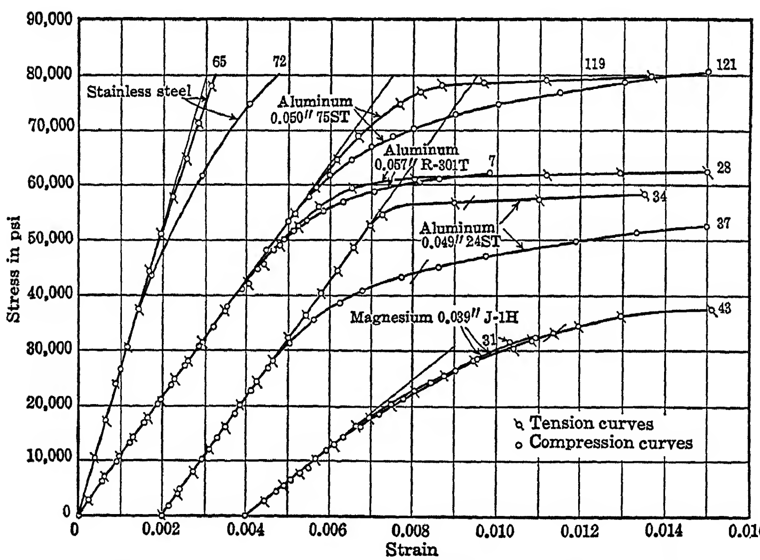


FIG. 6. Stress—Strain Curves of Sheet Metals Tested *in* Direction of Rolling.

FIG. 7. Bend Tests of Various Structural Metals Showing Differences in Ductile Properties.

Of recent years the word plasticity has been commonly used in the sense in which we use ductility. This, to me, suggests a direct translation of the German word "Plasticität." It may well be that the words "Dehnbarkeit" or "Zähigkeit" in the German language do not have the identical connotation of our word "ductility." That, however, is their, not our, worry. We see no reason for using the same word to describe the property of putty and that of metals. We still prefer the time-honored word ductility.

Strain Hardening and Strain Weakening. Strain hardening is another term which appears to be variously interpreted. We may be reasonably certain that, in the minds of some, plasticity means the characteristic property of mild steel as well as that of putty. As to strain hardening we confess to being in the dark as to the various shades of meaning commonly given to the term. It appears necessary, however, to use the term, and we shall therefore give our own explanation and definition of it.

As a mild steel is strained to a point E (Fig. 1), and the straining conditions are then reversed, the stress–strain relationship for the decreasing strains and stresses is represented by the curve EF. The material now is a cold-worked steel, either cold-stretched or cold-compressed. If the stress–strain relationship for this specimen is next ascertained, in the sense of cold working, then the stress–strain curve appears as $FE'G$. This is exemplified by curve 171 (p), Fig. 2, and curve 176(m), Fig. 3. If the specimen, instead of being tested immediately after the first cold working, is heated to the mild heat of 100 C and then tested in the same sense, its stress–strain properties are as represented by curve $FB''G'$, Fig. 1. This curve is exemplified by curve 172(p), Fig. 2, and curve 175(m), Fig. 3. (The letters p or m indicate the bars from which specimens were cut.) One phenomenon not shown on Fig. 1, but of considerable interest, is that a cold-worked mild steel, if allowed to rest at room temperature, has a tendency to restore its elastic properties in a manner similar to the restoration accomplished by mild heat applied for only a few minutes. Compare curves 171 (p), 78(d) and 172(p) in Fig. 2.

If the straining conditions are reversed after cold working, that is, if a cold-stretched steel is tested immediately afterward in compression, then the stress–strain relationship appears as FE'', Fig. 1, and the material may be said to be pronouncedly strain-weakened. This is exemplified by curve 114(t), Fig. 2, by curve 180(w), Fig. 3, and by curve 146(v), Fig. 4. The effect of aging or

the application of mild heat on the stress–strain properties, in a sense reverse to that of cold working, is similar to but more pronounced than in cold working and testing in the same sense. Mild heating after cold working improves the elastic properties in a sense opposite to that of cold working (elastic properties in compression after cold working in tension) as shown by curve FB', Fig. 1. See curves 114(t), 115(t), and 112(t), Fig. 2; curves 180(w), 178(w), and 177(w), Fig. 3; and curves 146(v) and 147(v), Fig. 4.

This improvement of the elastic properties of cold-worked metal, this restoration of the original value for the modulus of elasticity, this raising of the elastic-limit stress, as the result of aging or the application of mild heat (for steel 100 C or more, but less than the critical temperature), we define as strain hardening.

In "The Effects of Cold Working on the Elastic Properties of Steel" (reference 1a), the following five laws were formulated:

1. When mild steel is cold-worked and properly aged, or tempered and subsequently tested in the same sense as that of the cold working, its elastic limit may be raised more than 100 per cent, and from 10 to 20 per cent beyond the stress at which cold working was discontinued.
2. When mild steel is cold-worked in one direction and properly aged and tempered, though tested in either one of two senses of a different direction (for example, cold-stretched and aged and subsequently tested in either positive or negative torsion), then its elastic limit may be raised some 50 per cent.
3. When mild steel is cold-worked in one direction and sense and properly aged or tempered, though tested in the same direction but opposite sense, then the elastic limit remains at the value of the original elastic limit, but the yield point is raised.
4. When mild steel is cold-worked in any direction or sense, and tested in any direction or sense without any aging or tempering, then the elastic limit falls below the value of the original elastic limit, often down to zero.
5. Tempering cold-worked steel at temperatures from 100 to 300 C or aging cold-worked steel has a tendency to perfect its elastic properties. Tempering merely accelerates the effects of time.

We would now like to add a sixth law, a law which was proved at the time but not formulated.

6. The effects of cold working on the elastic properties of steel are a function of the strains involved in the cold-working processes and are independent of the stresses involved in these processes.

Figures 8, 9 and 10 show that the effects of cold working, the effects of aging and of the application of mild heat on the cold-worked material, such as 0-1 HTA magnesium and 24 ST aluminum extrusions, are similar to those experienced with steel. It thus

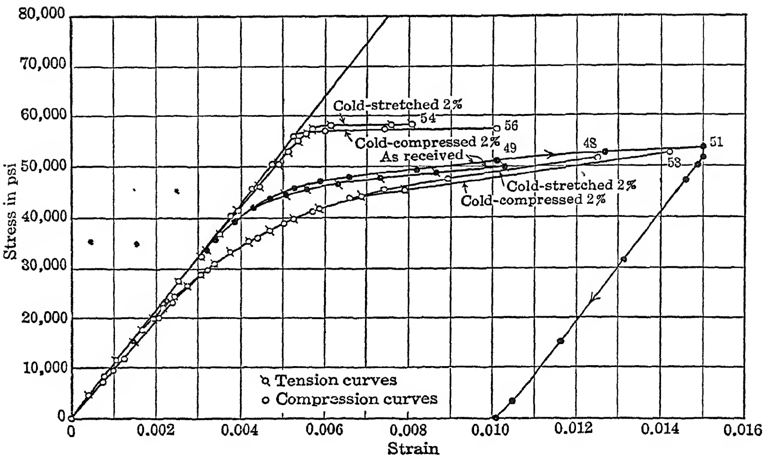


FIG. 8. Compression and Tension Stress–Strain Curves of 24 ST Aluminum-Alloy Extrusion. Showing effects of cold working.

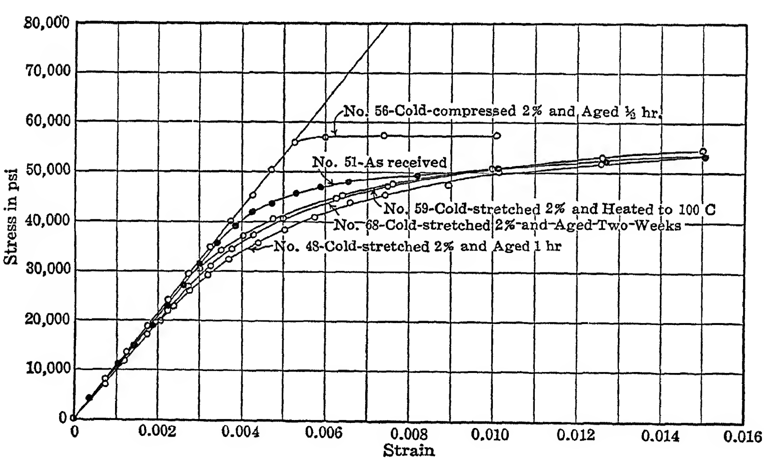


FIG. 9. Compression Stress–Strain Curves of 24 ST Aluminum-Alloy Extrusion. Showing effects of + and − cold working, of aging, and of heating to 100 C.

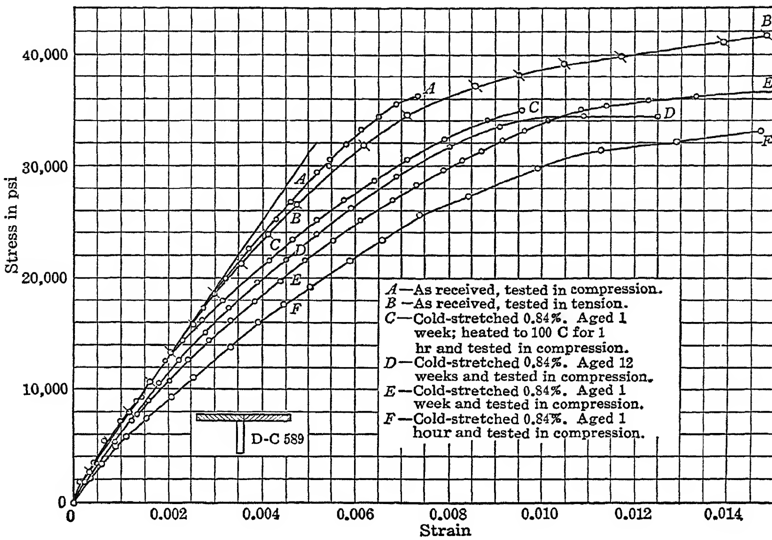


FIG. 10. Compression and Tension Stress–Strain Curves of 0-1 HTA Magnesium Alloy Extrusion. Showing effects of + and − cold working, of aging, and of heating to 100 C.

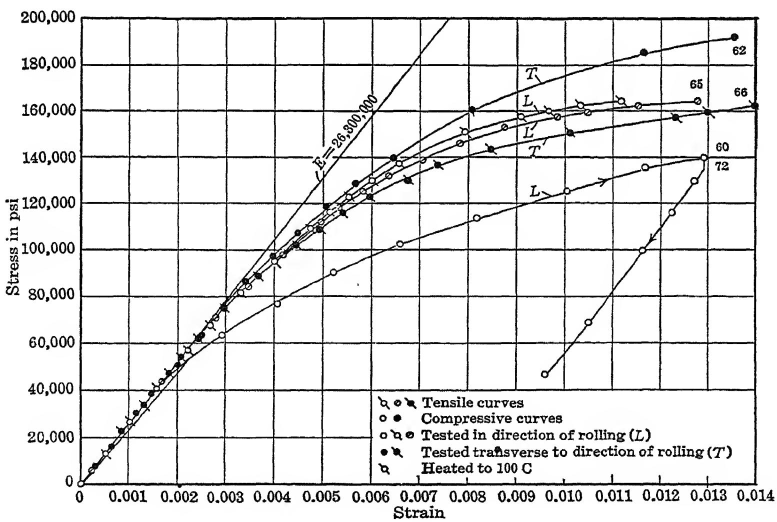


FIG. 11. Stainless Sheet Steel—Tension and Compression Stress–Strain Curves.

appears that in the foregoing six laws the words "24 ST aluminum extrusion" and "0-1 HTA magnesium extrusion" may be substituted for the word "mild steel" without doing violence to the truth (see references 1*i* or 1*m*).

One misconception, commonly encountered, with reference to cold-working effect is the belief that cold working results in strain hardening (raising of the elastic limit) and is therefore beneficial. As we have tried to show, there is strain weakening (in a sense opposite to that of cold working) as well as strain hardening in the sense of cold working. Stringers may be bent cold in order that they may be assembled in airplane construction. The possibility of heating them, after being cold-worked, thus is simply nonexistent. Aging before the airplane is completed will partially alleviate the effect of cold working, but it will never completely annul the strain-weakening effect.

Figure 11 shows the effect of commercial cold rolling on the directional strength properties of stainless steel. Note how the compressive-strength properties in the direction of rolling are relatively depressed.

Simple cold stretching, or cold compressing, does not result in residual stresses. Cold bending, however, does result in residual stresses. Such residual stresses materially complicate the analysis of strain-hardening or strain-weakening effects. They are discussed on page 23.

Chapter 2

LIMIT DESIGN OF SIMPLE STRUCTURES

Sense of Value. In the preface we have emphasized the importance we attach to sense of value, to premises, and to assumptions, in the formulation of any theory. To these may be added the definitions of words and phrases. It does not matter what we call a thing or an idea, but it matters a great deal that the words used to give expression to a concept or an idea held by the writer convey a closely similar concept or idea to the reader. In engineering discussions we are, unfortunately, prone to use words and terms often of ambiguous and even contradictory meaning. Commonly we call a treatise theoretical when we mean mathematical. The two terms are not intrinsically synonymous. The photoelastician talks of nothing but stress analysis, when, we believe, he observes nothing but strains. Similarly, the members of the Society for Experimental Stress Analysis, to our knowledge, never measure stresses; they measure only loads and deformations, principally the latter. It can be argued that the deformations may be interpreted as stresses. This does not alter the fact that the strains are primary and the stresses secondary, and in the transition from the one to the other there frequently is a great deal of room for difference of opinion.

In my own book, *Elastic Energy Theory* (reference 1*h*), which is a fairly comprehensive treatise on the theory of elasticity as applied to the problem of strength, I called attention to the fact that the theory of elastic energy, as well as the entire mathematical theory of elasticity, is but a supplementary theory. I did this, not to belittle my own book or the mathematical theory of elasticity, but to be honest with myself as well as with my readers.

The theory of strength is predicated on two basic considerations, first, that of equilibrium, and second, that of continuity. The first is by far the more important one of the two. It is expressed by Newton's laws of motion, $F = ma$, or, when the acceleration is zero, as $\Sigma F_x = 0$, $\Sigma F = 0$ and $\Sigma M = 0$. When the laws of equilibrium

are violated by only so much as a hairbreadth, it spells collapse. The considerations of continuity stipulate that no cracks occur, or that no sudden and excessive deformations take place. To satisfy considerations of continuity we have a choice of two theories, the theory of elasticity or the theory of limit design. Either of these theories supplements the theory of static equilibrium in arriving at a picture of strength.

The Simple Tension Bar. A bar subject to an axial load (Fig. 12*a*) is one of the simplest structures we have. Considerations of

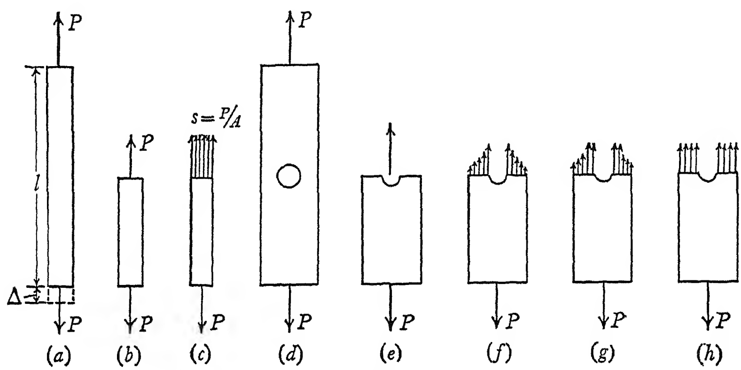


FIG. 12.

static equilibrium tell us (Fig. 12*b*) that on any transverse plane the resultant force is P (ignoring the weight of the bar). Our most elementary formula in strength of materials tells us (Fig. 12*c*) that the stresses are

$$s = \frac{P}{A} \qquad \text{Formula I}$$

To arrive at formula I, we require something more than considerations of equilibrium. We must make some assumptions as to continuity and from these deduce certain conclusions as to stress distribution. We may resort to the theory of elasticity and argue as follows: If the load is perfectly concentrically applied, if the bar is of uniform cross-section area, perfectly homogeneous, free from imperfections and scratches and free also from any residual stresses, then the strains may reasonably be assumed to be uniform over any cross-section area. Then, if the material is elastic, the stresses will be uniformly distributed, and $s = P/A$. The theory of limit design argues as follows: Slight eccentricity in the

application of the load, or slight imperfections or scratches, do not materially affect the limit stress distribution; neither does the presence of residual stresses, provided the material is ductile. If the material is ductile, then, before rupture or excessive deformation can take place, the stress distribution will be as shown in Fig. 12*c*, and formula I will be justified. Intrinsically, we care nothing about what the stresses in the bar may be under intermediate conditions of loading. We are essentially concerned in avoidance of either rupture or excessive deformation, and formula I is a good measure of the beginning of excessive deformations, according to the philosophy of limit design.

In regard to a bar with a hole punched through it, say in order to effect a rivet connection, the assumption that strains are uniformly distributed along the horizontal diameter passing through the hole (Figs. 12*d* to 12*h*) is definitely not justified. If perfect homogeneity, absence of scratches, residual stresses and perfect concentricity in the application of the load are assumed, then the theory of elasticity concludes that the stresses near the horizontal diameter and adjacent to the hole may be some 200 or 300 per cent in excess of the average stresses, which stresses are the load divided by the net area. The theory of limit design, disregarding residual stresses, scratches, holes and slight eccentricities in the application of the load, concludes that, before excessive deformation or rupture can take place, a stress distribution as shown in Fig. 12*h* will be realized and thus may be used to serve as a guide to prevent such excessive deformation or rupture.

The Rivet Connection. The simple rivet connection presents a highly complex problem, if viewed from the elasticity point of view, because of the assumptions which we are forced to make, as

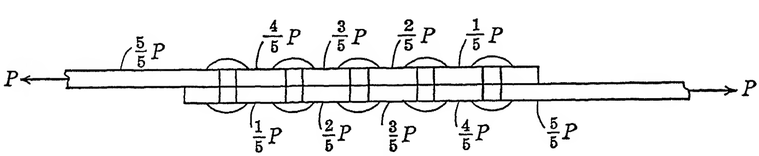


FIG. 13.

to alignment of rivets and friction between plates which represent conditions that may not be similar in any two rivet connections. It is commonly assumed that each rivet transmits its share of the load. If this were true, then the portion of the load carried by the plates would be as indicated in Fig. 13. This obviously

conflicts, in case of purely elastic behavior, with the assumption that the alignment of the rivets remains substantially unchanged, since, of two adjacent plates, one would be strained four times as much as the other. There is not much wrong with the assumption, we believe. The theory of elasticity concludes that the outer rivets carry the major share of the load and that the inner three are practically duds. This is beautifully verified by laboratory tests (reference 4). Differences of opinion, if any, derive from differences in sense of value. The theory of limit design argues that, if the material is ductile, we have no interest in stress distribution within the elastic range. We are primarily concerned that the rivet connection shall not rupture or deform excessively. If it is assumed that the connection is designed so as not to fail by tearing through the plate, the outer rivets will, under progressively increasing loads, undoubtedly be first to reach the yield stress. From then on, if the loads to be transmitted continue to increase, the portion of the load transmitted by the two outer rivets will remain substantially constant, while the stress on the inner rivets is gradually increased. If there are not too many rivets in a row, then the portion of the load transmitted by each rivet will be equalized before rupture, or excessive deformation, takes place.

It is now seen that neither formula I, $s = P/A$, nor the basic rule relative to design of rivet connections, can be justified by elasticity considerations. Both of them are justified by considerations of limit design.

Freedom from scratches, freedom from residual stresses, and near-perfect alignment were the assumptions necessary to justify formula I on the basis of elasticity considerations. To justify it on the basis of limit design none of these assumptions is required. A scratch represents a discontinuity and acts as a stress raiser—witness Fig. 12*f* and the practice of a worker in glass who uses a diamond cutter purposely to obtain a rupture at a predetermined point. A worker in steel, on the other hand, scratches his material with a scratch-all with impunity. A scratch, from the point of view of limit design, would decrease the net cross-section area by the depth of the scratch, involving an error of secondary order of magnitude, while within the elastic range the increase in stresses around the groove might easily amount to several hundred per cent. As to residual stresses, they seem to be almost universally and continually ignored. Yet they are practically always present. As

to alignment, one need but ask a laboratory technician to learn how difficult it is to obtain it, and to realize that we are not justified in assuming its existence in practice.

The Simple Cantilever Beam. If a simple cantilever beam of elastic–ductile material, say structural steel, be loaded with a concentrated load at the end, Fig. 14*a*, the stress distribution over

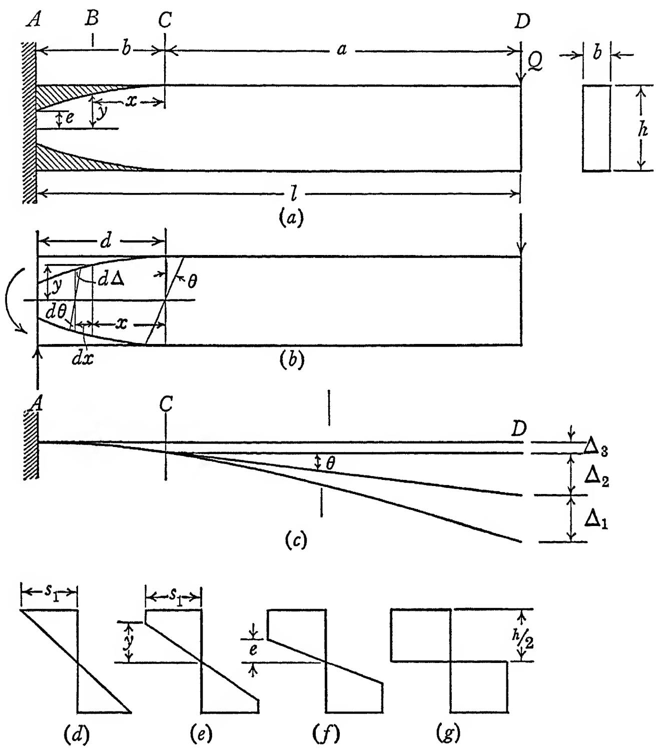


FIG. 14.

any cross section will at first be linear, Fig. 14*d*. If the tensile and compressive elastic–ductile properties of the material correspond to curve *ABD*, Fig. 1, further, if a plane before bending remains a plane during bending in any elastic part of the beam, then the stress distribution over any cross section of the beam, which is partially strained beyond the elastic limit, will be as shown in Fig. 14*e*. Figure 14*a* represents a cantilever beam of rectangular cross-section area *bh*. The shaded area represents

the portion of the beam strained beyond the elastic limit. Figure 14*d* represents stress distribution at section C, Fig. 14*e* that at section B, and Fig. 14*f* that at section A.

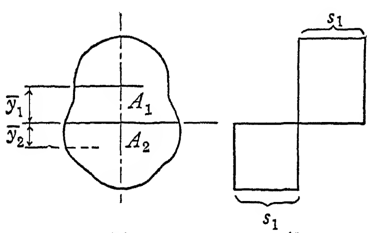
The maximum value of the load Q, for which the elastic-limit stress is just reached in the outer fibers at the built-in end, is found from the equation:

$$s_1 = \frac{Mc}{I} = \frac{6Q_1l}{bh^2}$$

From which

$$Q_1 = \frac{s_1bh^2}{6l} \qquad \text{Equation a}$$

The limiting resisting moment of the beam will be developed when all the redundant fibers over the entire cross section are stressed with their elastic limit stress s_1. The stress distribution over the cross section of the beam will then be as shown in Fig. 14*g*.



(*a*) Cross Section (*b*) Limit-Stress Distribution

FIG. 15.

Figure 15*a* represents a cross section of a beam unsymmetrical about one axis in which the limit stress distribution has been reached. From the equation, $\Sigma F_x = 0$, we derive $A_1s_1 = A_2s_1$. Thus

$$A_1 = A_2$$

Therefore it appears that the neutral axis has shifted. It no longer passes through the centroid of the cross section.

The bending moment resulting from a stress distribution, such as shown in Fig. 15*b*, would be: $M = s_1 (A_1\bar{y}_1 + A_2\bar{y}_2)$. In the case of a symmetrical beam the neutral axis remains in place, passes through the centroid of the cross section, and $A_1\bar{y}_1 = A_2\bar{y}_2$. In this instance the preceding formula may be written

$$M = 2s_1A\bar{y} \qquad \text{Formula II}$$

In this formula $A\bar{y}$ represents the static moment about the neutral axis of a part of the cross-section area which lies to one side of the neutral axis.

Thus for the rectangular beam under consideration the limit bending moment M_3 is

$$M_3 = Q_3 l = 2s_1 \frac{bh}{2} \times \frac{h}{4} = \frac{s_1 bh^2}{4}$$

and

$$Q_3 = \frac{s_1 bh^2}{4l} \qquad \text{Equation b}$$

Thus the limit load-carrying capacity of the rectangular cantilever beam is 50 per cent greater than its elastic load-carrying capacity.

The total linear displacement of the end of the cantilever (Fig. 14*c*) may be expressed as the sum of three linear displacements, namely,

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3$$

in which Δ_1 is the vertical displacement of point C resulting from the curvature of the elastic portion of the beam, or

$$\Delta_1 = \frac{Qa^3}{3EI}$$

Δ_2 is the displacement of point D resulting from the deflection of the tangent at point C, or

$$\Delta_2 = a\theta \qquad \text{Equation c}$$

and Δ_3 is the vertical displacement of point C, or

$$\Delta_3 = \int_0^d x\, d\theta \qquad \text{Equation d}$$

The ductile portion of the beam does not affect the values of Δ_2 and Δ_3 other than that it determines the equation of the boundary curve, $y = f(x)$. Once the boundary curve $y = f(x)$ is known, we may write

$$\theta = \int d\theta = \int \frac{d\Delta}{y} = \int \frac{s_1\, dx}{Ey} \qquad \text{Equation e}$$

Here s_1 is elastic limit stress. From this we obtain both Δ_2 and Δ_3.

This boundary curve, $y = f(x)$, is found as follows: The stress in the outer fibers at cross section C, a distance a from the free

end, is equal to the elastic limit stress s_1. The distance a is found from the elasticity equation,

$$s_1 = \frac{Mc}{I} = \frac{6Qa}{bh^2}$$

or

$$a = \frac{s_1 bh^2}{6Q} \qquad \text{Equation f}$$

The maximum bending moment at the point of support (point A) is Ql. Considering the stress-distribution diagram shown in Fig. 14f, we may write

$$M = Ql = \frac{s_1 bh^2}{4} - \frac{s_1 be^2}{3}$$

from which we obtain

$$e = \sqrt{\frac{3h^2}{4} - \frac{3Ql}{s_1 b}} \qquad \text{Equation g}$$

The bending moment at point B (Fig. 14a) is

$$M_b = Q(a + x) = \frac{s_1 bh^2}{4} - \frac{s_1 by^2}{3} \qquad \text{(See Fig. 14}e\text{)}$$

Substituting equation (f) for a, we obtain

$$x = \frac{s_1 bh^2}{4Q} - \frac{s_1 by^2}{3Q} - \frac{s_1 bh^2}{6Q} = \frac{s_1 b}{12Q}(h^2 - 4y^2)$$

$$\frac{dx}{y} = \frac{2s_1 b}{3Q}(-dy) \qquad \text{Equation h}$$

$$\frac{x\,dx}{y} = \frac{s_1{}^2 b^2}{18Q^2}(-h^2 + 4y^2)\,dy \qquad \text{Equation i}$$

Substituting equation (h) in equations (c), (e), and (f), we obtain

$$\Delta_2 = \frac{s_1 a}{E}\int_{x=0}^{x=d} \frac{dx}{y} = \frac{s_1{}^2 bh^2}{6QE}\int_{y=\frac{h}{2}}^{y=e} \frac{2s_1 b}{3Q}(-dy) = \frac{s_1{}^3 b^2 h^2}{9Q^2 E}\Big[-y\Big]_{\frac{h}{2}}^{e}$$

$$\Delta_2 = \frac{s_1{}^3 b^2 h^2}{9Q^2 E}\left(\frac{h}{2} - e\right)$$

Substituting equation (i) in equations (e) and (d), we obtain

$$\Delta_3 = \frac{s_1}{E}\int_0^d \frac{x\,dx}{y} = \frac{s_1}{E}\int_{\frac{h}{2}}^e \frac{s_1{}^2b^2}{9Q^2}\left(2y^2 - \frac{h^2}{2}\right)dy$$

$$\Delta_3 = \frac{s_1{}^3b^2}{9Q^2E}\left[\frac{h^3}{6} - \frac{h^2e}{2} + \frac{4e^3}{6}\right]$$

Substituting equation f in the expression for Δ_1 we obtain

$$\Delta_1 = \frac{Qa^3}{3EI} = \frac{s_1{}^3b^2}{9Q^2E}\left(\frac{h^3}{6}\right)$$

The total linear displacement at the free end is

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 = \frac{s_1{}^3b^2}{9Q^2E}\left(\frac{5h^3}{6} - \frac{3h^2e}{2} + \frac{4e^3}{6}\right) \qquad \text{Equation j}$$

The maximum purely elastic deflection is

$$\Delta_1 = \frac{Q_1l^3}{3EI} = \frac{2s_1l^2}{3Eh} = \frac{18s_1l^2}{27Eh} \qquad \text{Equation k}$$

The limit deflection, when e just becomes zero, from Equation (j), s

$$\Delta = \frac{5}{54}\frac{s_1{}^3b^2h^3}{Q_3{}^2E}$$

Substituting Q_3 from Equation (b), we have

$$\Delta = \frac{40s_1l^2}{27Eh} \qquad \text{Equation l}$$

To find the displacements for loads intermediate between Q_1 and Q_3 (equations k and l), it is necessary only to substitute in equation (j) the desired value for Q, and the corresponding value or e as found from equation (g), and solve.

Figure 16 represents three graphs, showing the deflections of antilever beams after the elastic-limit strain has been exceeded Fig. 16a, the graph for a beam of rectangular cross section; Fig. 6b for a beam of circular cross section; and Fig. 16c for a 4 x 6 in. VF 12-lb beam). The open circles represent experimental values; he closed circles represent analytical values computed as outlined reviously.

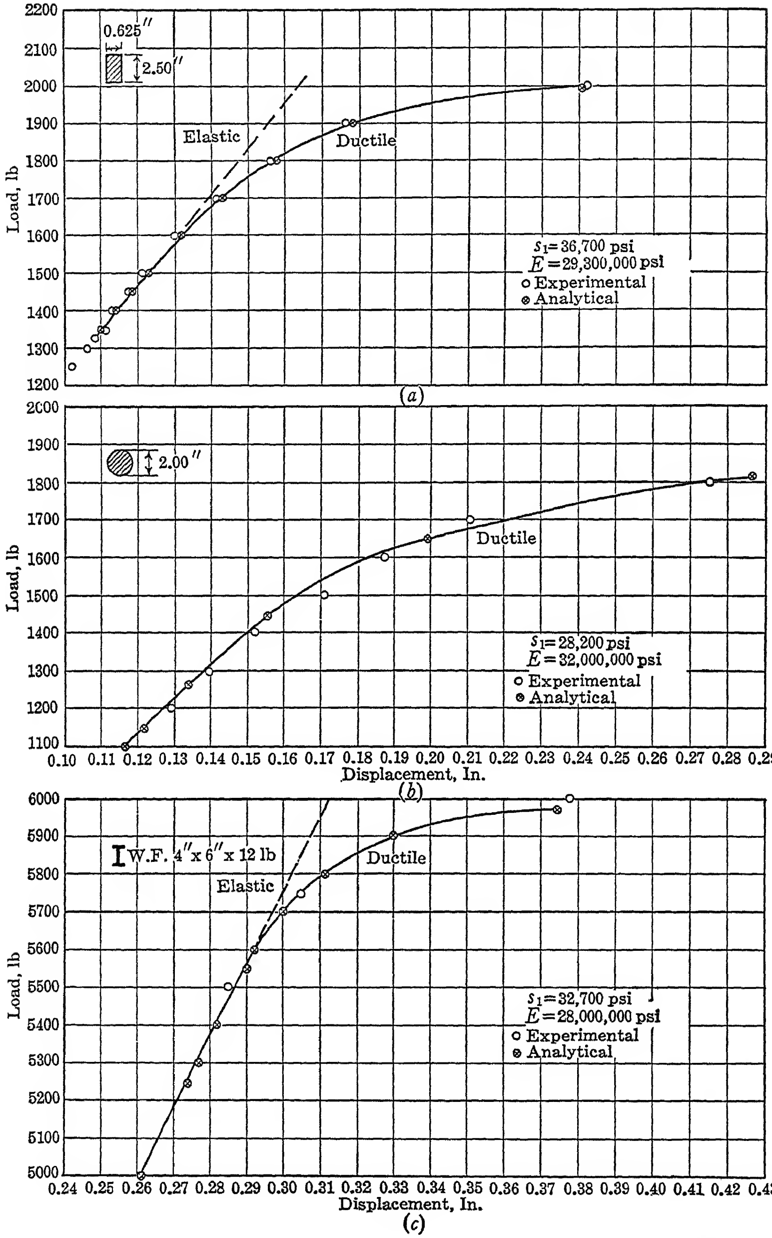


FIG. 16. Experimental and Theoretical Load-Deflection Curves of Steel Cantilever Beams Loaded Beyond Elastic Limit.

The foregoing discussion on the behavior of simple cantilever beams, loaded beyond the elastic limit, has been largely taken from a doctoral thesis submitted to the University of Michigan in 1939 by Dr. E. O. Scott, professor of engineering mechanics, of Toledo University (reference 5). Figure 16 is also reproduced from this thesis.

Residual Stresses. Assume the elastic–ductile stress–strain relationship of the material of which a beam is made to be closely represented by curve $ABCC'$ (Fig. 1). When the beam is first loaded to its full capacity, limit load, corresponding to the stress pattern of Fig. 17*c*, the stress–strain relationship for this increasing load is then represented by curve ABC of Fig. 1. If the load is next decreased, the stress–strain relationship for this decreasing load is represented by curve CC' of Fig. 1. In other words, if the beam has been overstrained and the load is removed, then, under the decreasing load, the beam functions elastically. For an experimental record of this phenomenon see curve 10 in Fig. 5.

Consider a beam of rectangular cross section bh (Fig. 17*a*) loaded with a moment M_1, which stresses the extreme fiber in the beam up to the elastic limit s_1. M_1 is then equal to $s_1bh^2/6$. If the elastic limit in the top and bottom fibers of the beam is exceeded, then the stress distribution over the cross section of the beam appears as shown in Fig. 17*b*, and the magnitude of M_2 is expressed as

$$M_2 = s_1 b \left(\frac{h^2}{4} - \frac{y^2}{3}\right)$$

When the limit moment is applied so that all the fibers over the entire cross section have become ductile (Fig. 17*c*), then y in equation (m) equals zero, and we obtain

$$M_3 = \frac{s_1bh^2}{4}$$

To remove the limit load M_3 is equivalent to superimposing, upon the load shown in Fig. 17*c*, a load of equal magnitude but of opposite sign such as the load shown in Fig. 17*e*. If, upon being unloaded, the material behaves elastically as was assumed, then the stress distribution over the cross section of the beam, to be superimposed on the existing stress distribution of Fig. 17*c*, appears as shown in Fig. 17*e*. Superimposing the load and stress conditions of Fig. 17*e* on those of Fig. 17*c*, we obtain the condition

represented by Fig. 17g. That is, there are no external loads acting on the beam, the residual stresses at the extreme fiber are of a magnitude $0.5s_1$ and those at the neutral axis are of a magnitude s_1. It may readily be observed that the sum of all the forces over

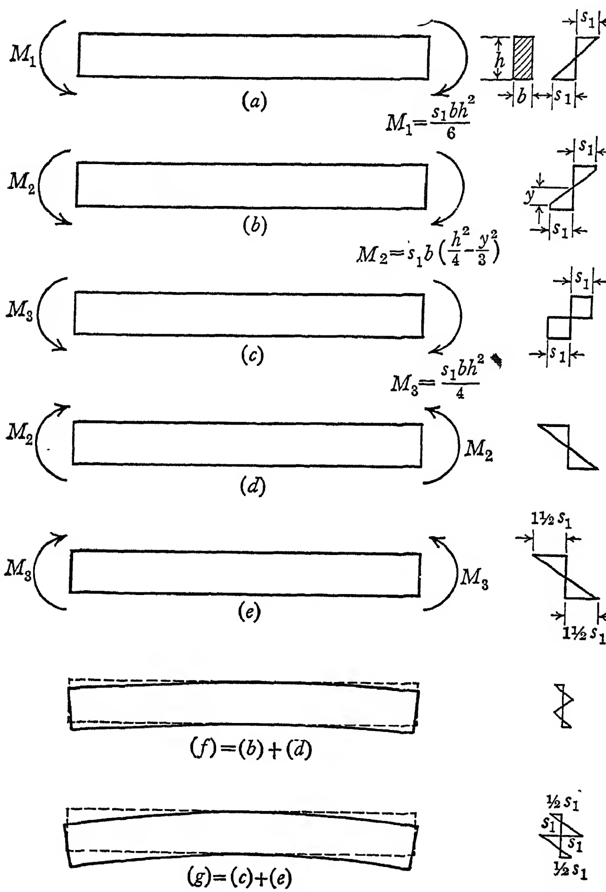


Fig. 17.

the entire cross-section area (Fig. 17g), as well as the sum of the moments of these forces, is zero.

If some of the material on the outside should be removed, then the moments due to the residual stresses no longer balance, and a change in curvature results. This explains why steel shafts, or beams which were straightened cold by bending, warp when some

of the outer material is removed either by machining or as the result of wear.

When a beam is strained beyond the elastic limit with a moment M_2, so that $M < M_2 < M_3$, and this load M_2 is subsequently removed, then the resulting beam and residual stresses are as represented by Fig. 17*f*. If a beam, once strained beyond the elastic limit and with the load removed, is subsequently loaded with a load of the same magnitude and sense, then no further ductile yielding takes place, and the stress distribution over the cross section of the beam is as that represented by either Fig. 17*b* or 17*c*. For example, let us reverse the arrows representing the loading in Fig. 17*e*, and correspondingly reverse the stress diagram of Fig. 17*e*. If next we superimpose that loading and stress condition on the condition shown in Fig. 17*g*, then we obtain, without further exceeding the elastic limit, the loading and stress condition shown in Fig. 17*c*.

Reversal of Stresses. If, on the other hand, a beam, once strained and with the load removed, is subsequently loaded with a load of opposite sense, then the residual stresses are unfavorable, and the elastic limit is reached sooner than it was in the first instance. For example, if a moment of magnitude $M_1/2$, but of a sense opposite to that shown in Fig. 17*a*, is applied to the beam shown in Fig. 17*g*, then the elastic limit will be reached at the extreme fibers. If any greater moment is applied, it will result in additional ductile yielding.

In case of reversed loading, if further ductile yielding is to be avoided, *the sum of the positive and negative loading should not exceed twice the amount of the maximum elastic loading.* If it does exceed this maximum, ductile yield takes place after each cycle of loading, and rupture results after a few such cycles. This is like the procedure we employ when breaking a wire, by means of clamping it in a vise and bending it in opposite senses for a number of times.

Springs. An interesting example of prestressing a structure, so that it functions with highest efficiency, is represented by the manufacture of spiral springs. A spiral spring is made of flat stock of a high-grade alloy steel. The final operation in the manufacture is nothing more than tightly winding a flat strip of steel on an arbor. This operation gives the flat strip a permanent set and causes it to assume, when released, the shape of the spiral. If it is released, the residual stresses in the spring are as repre-

sented by Fig. 17*g*. The chances of subsequently loading a power spring in reverse sense are generally precluded. If it is subsequently loaded in a sense similar to the loading involved in the manufacture, a stress distribution similar to Fig. 17*c* results. Such a stress distribution is equivalent to the highest obtainable efficiency in the storage of energy and results in an expression for elastic energy: $s_1{}^2V/2E$, in which V represents volume of spring. If the stress distribution over the cross-section area of a spring of rectangular cross section were as represented by Fig. 17*a*, the energy in the spring would be given by the expression: $s_1{}^2V/6E$, or only 33 per cent as much.* A spiral spring is a self-locking mechanism and, once made, cannot be overstressed. The ductility of the material is very slight indeed. The elastic limit and ultimate strength are practically the same. The working stresses, therefore, are very high, almost as high as the ultimate stresses. Working stresses of a magnitude of 250,000 lb per sq/in. are obtainable (see reference 1*l*).

The spiral spring offers an excellent example of a principle which we propose to illustrate further by means of other examples, to the effect that *loading a redundant structure beyond the elastic limit automatically stores residual stresses in the structure of the right magnitude, and of the correct sign, such that, under subsequent loading of the same magnitude and sense as that involved in the original loading, the structure behaves purely elastically.*

Evaluation of Assumptions. We have previously asserted our belief that in natural philosophy the premises, assumptions, and sense of value are of primary importance, and that all logic built on this sense of value and on these assumptions and premises is secondary. No assumptions can be perfect and they remain open to doubt until substantiated by tests or by experience.

When the elastician or stress analyst assumes homogeneous material free from residual stresses, he assumes something which is nearly impossible to realize. He is saved from making a fatal blunder not because of any elasticity reasoning, but because of the logic of limit design which argues that, provided there is a measure of ductility, the limit-stress pattern is not, or but very slightly, affected by any residual stresses which may initially be present.

The assumed tension and compression stress–strain relationship (curve ABC, Fig. 1), upon which the ductile behavior of cantilever

* Only 75 per cent of the energy represented by $s^2V/2E$ is available upon release of the spring.

beams is predicated, seems quite well substantiated—first, by curve 118(t), Fig. 2, curve 179(w), Fig. 3, and curves 10 and 18 (Fig. 5); and, second, by the quite acceptable agreement between analytical and experimental values as they appear in Fig. 16. However it seems unreasonable to assume that the elastic core near the neutral axis ever vanishes completely. If the distance $A'D = 20\ A'B$ (Fig. 1), then the limit stress pattern would appear as shown in Fig. 18a rather than as that shown in Fig. 14g. The

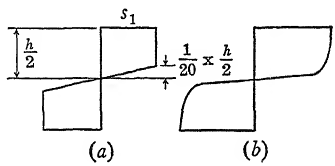


FIG. 18.

difference between the two figures would have but an extremely small effect on the corresponding resisting moment. If the material were stressed beyond point D (Fig. 1), then the limit-stress pattern might appear as Fig. 18(b). Such excessive deformation, for the present, holds no interest in theory of limit design. If it did, the corresponding changes in dimensions of the cross section of the beam would have to be evaluated, which would complicate matters still further.

In regard to the assumed stress–strain relationship (curve CC', Fig. 1), upon which the residual-stress analysis is predicated, we are still far from being on solid ground. For mild steel, aluminum, or magnesium, the stress–strain relationship after cold working, in a sense opposite to that of cold working, is more like curve EFE'' than like CC'. Our conclusions concerning residual stresses should thus be taken as a first approximation to a probable truth.

Chapter 3

LIMIT DESIGN OF REDUNDANT BEAMS

Redundancy. A redundant structure is one that contains more members, reactions, or restraints, and thus, in mathematical parlance, more unknowns, than are necessary to satisfy the conditions of equilibrium. The reactions of the simple beam (Fig. 19*a*) are statically determinate. This is not true, however, in case of the built-in beam (Fig. 20). As the result of symmetry considerations we may conclude that $R_1 = R_2 = wl/2$, but we are left with an unknown reaction, M_a, for which there is no independent equation available within the laws of equilibrium. These equations of equilibrium need supplementing; hence the term, "statically indeterminate." The elastic curve for the built-in beam has horizontal tangents at points A, C, and E (Fig. 20*b*); thus of necessity we have two points of inflection, points B and D. This elastic curve, therefore, represents a combination of both simple-beam and cantilever-beam action; hence the terms "statically indeterminate" and "redundant" are synonymous terms. Either simple-beam action alone, or cantilever-beam action alone, would satisfy equilibrium requirements. When we have a combination of the two we need supplementary independent equations, derived from a theory which is independent of the laws of static equilibrium, to effect a complete solution. Both the theory of elasticity and the theory of limit design offer such a supplementary theory. Which of these two theories one might prefer depends very largely on one's sense of value.

The shear diagram for the simple beam (Fig. 19*c*) is identical with the shear diagram for the built-in beam (Fig. 20*c*). From the formula, $dM/dx = V$, or $M = \int dM = \int V\,dx$, it follows that, since the shear diagram for the two beams are identical, their bending-moment diagrams must be identical, except for the constant of integration. Determination of the constant of integration places the X axis, with reference to which the bending-moment

diagram is drawn, so as to make the bending moments at the ends in one instance equal to zero, and in the other, equal to M_a. In both cases $M_a + M_c = wl^2/8$. However, since $M_a = 0$, in the simple beam, $M_c = wl^2/8$. In the case of built-in beams we need a supplementary equation before we can determine the values of M_a and M_c.

We are concerned with theory, and we aim to emphasize what we regard as the most important aspects of any theory, namely, sense of value and assumptions. The formula, $dM/dx = V$, is an equilibrium equation. Thus our conclusion

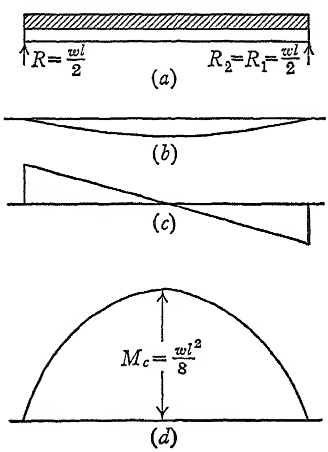


FIG. 19.

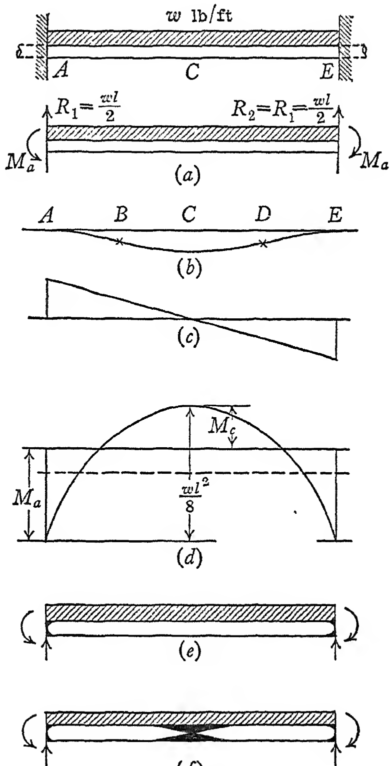


FIG. 20.

that $M_a + M_c = wl^2/8$ is one derived from equilibrium considerations. We can claim very little "exactness" in engineering. But the conclusion that $M_a + M_c = wl^2/8$ we do claim as being exact, simple, and primary. Any theory we might employ to determine the required independent relationship between M_a and M_c will be not only second in order, but also quite secondary in value, supplementary, and sometimes quite mathematically involved. The answer provided by the theory of elasticity is that $M_a/M_c = 2$, and, thus, $M_a = wl^2/12$. The solution provided by the theory of

limit design is that $M_a/M_c = 1$, and therefore $M_a = wl^2/16$. Which of these answers we prefer will depend on what we mean by strength, on our objective, and on our sense of value.

Definition of Structural Strength. It seems that we have emerged from an era of statically determinate construction into that of redundant construction. However, certain conventions, rules, practices and a certain sense of value, developed during the former period have been carried over into the latter in which certain of these rules and values are definitely open to question.

Failure in structural engineering is not a question of rupture. A steel beam can be ruptured only if we shear it, saw it, or burn it in two. If a structure fails, its beams will be bent and twisted somewhat in the manner of lead pipes, but they will not rupture. Rupture when it occurs generally takes place after failure. It is generally an effect rather than a cause and then occurs only as a consequence of an inadequately designed connection detail.

In a statically determinate truss each member has a definite task to perform. If the elastic-limit stress in one member is reached (point B, Fig. 1), then excessive deformation (BD, Fig. 1) takes place under a constant loading. This excessive deformation generally spells doom to the structure. In order to avoid it, a working stress, a fraction of the elastic-limit stress, arrived at by dividing the latter by a factor of safety, was introduced. Although by this procedure the emphasis was shifted from deformation to stress, it does not alter the fact that the primary criterion of strength was and is deformation.

In the simple beam (Fig. 19*a*), in which the reactions though not the stresses, are statically determinate, a similar situation prevails. Once the elastic-limit stress in the middle of the beam is reached, ductile yielding commences. This ductile yielding does not become excessive, or uncontrollable, until the ductile stress pattern (Fig. 14*g* or Fig. 17*c*) has been nearly developed. This ductile stress pattern means a resisting moment in case of beams of rectangular cross section 50 per cent in excess of the elastic resisting moment in rectangular beams. In the more common W.F. beams or I beams the limit resisting moment is only from 6 to 10 per cent in excess of the elastic resisting moment. By ignoring it the structural (civil) engineer erred on the safe side.

In redundant beams conditions are strikingly different. The essence of redundancy is that two or more members, reactions, or restraints function to the same end. Elastic criteria prescribe

one, limit design stipulates another relationship between the carrying capacities of component members of a structure.

The Built-In Beam. Figure 20*b* shows the built-in beam as composed of two cantilevers, *AB* and *DE*, and one simple beam, *BD*. The problem is to determine how much load the beam can carry. One way of phrasing that problem is to determine how much cantilever and how much simple-beam action there is. During the past 40 years I have become progressively less interested in elastic-stress analysis as a criterion of strength in structural engineering. The structural-engineering profession provides a rough rule to the effect that deformations are to be restricted to $\frac{1}{360}$ of the length. We prefer another criterion and propose in this book to use elastic deformations of simple structures of normal proportion as our yardstick of permissible deformations. Within the elastic range, until the elastic-limit stress is reached at point *A* (Fig. 20*a*) the ratio M_a/M_c is constant and equals 2. When the beam is loaded beyond the elastic-limit load, then a small portion of the beam at *A* and *E* becomes ductile (black areas Fig. 20*e*), and the resisting moment at point *A* increases slowly (say, up to 6 to 10 per cent in W.F. beams), while the bending moment at *C* increases more rapidly than before. Uncontrolled deformation does not take place until the elastic-limit stress is reached also at the midpoint of the beam, and the ductile material in the beam is distributed as shown by the black areas in Fig. 20*f*. When that condition is reached, the ratio M_a/M_c will no longer equal 2. It then equals unity. In terms of Fig. 20*d*, it means that the axis has been lowered to the position shown by the dashed line, which in turn means that the points of inflection, *B* and *D* in Fig. 20*b*, have moved farther apart, which further means that we have relatively less cantilever action and more simple-beam action in the built-in beam under limit loading than under elastic loading.

Our proposal, in case of the built-in beam, to make the limit loading or the equation $M_a/M_c = 1$, the basis of design, does not mean that we propose to exceed the elastic-limit stress. The elastic-limit stress will be exceeded under near-limit loads, but under working loads it will not be exceeded. In the days when the working-stress idea was developed by dividing elastic-limit stress by the factor of safety, one might have arrived at identical results, if, instead, the safe load had been multiplied by the factor of safety and the working stress defined as equal to the elastic-limit stress. It seems a pity this was not done, as then the transi-

tion from the elasticity-strength philosophy to that of the limit-design philosophy would have seemed less abrupt. Limit design proposes to multiply the working load by a factor of safety, or an overload factor, and to design the structure so as to fail under this limit load. The criterion which justifies this procedure was formulated in reference 1*b* as dictum 3:

> **When in a well designed, and especially a properly detailed, *n*-fold redundant structure *n* redundant members are stressed to their elastic limit or up to their critical buckling load, the deformations involved are of the order of magnitude of elastic deformations until an $(n+1)$th member has reached its elastic limit, or its critical buckling capacity.**

The built-in beam of Fig. 20 is twice redundant. The determination of one of the redundants by means of symmetry (the moment at E equals that at A) does not alter this fact. When the beam is stressed beyond the elastic limit at points A and E, an elastic core remains, as shown by the unshaded area in Fig. 20*e*. The deformations of this core are therefore of the order of magnitude of elastic deformations. In fact, under a near-limit load, a load which will start ductile behavior in the middle of the beam, the deformations of the built-in beam will be less than those of the simple beam under a capacity elastic loading (Fig. 19*a*).

The Steel Beam on Three Supports. An elastic continuous beam of two equal spans loaded with a uniformly distributed load (Fig. 22*a*) has, so long as it functions elastically, a bending-moment diagram, as shown by the short dashed line in Fig. 22*e*. The maximum bending moment between supports of this beam is $0.0703wl^2$, while the moment over the central support is $wl^2/8$ or $0.125wl^2$. This maximum bending moment over the support is thus equal to the maximum bending moment in a simple beam (Figs. 21*a* and 21*d*, reference 1*c*). If the elastic working stress is accepted as a criterion of strength, then the implication is unmistakably to the effect that the strength of the continuous beam (Fig. 22*a*) is identical with that of the simple beam (Fig. 21*a*). In effect, the elastic-working-stress concept carried to its logical conclusion argues that the continuous beam (Fig. 22*a*) may be replaced by two simple beams without its load-carrying capacity being affected. Not only is this the direct implication of the theory of strength of materials as discussed in the various textbooks from which I have taught for 35 years, but also it is the conclusion definitely given in such an authoritative handbook as the *United States Steel Pocket Companion*, third reprint, 1937, page 157. In

this handbook the safe load-carrying capacity of the simple beam (Fig. 21*a*) and that of the continuous beam of two equal spans (Fig. 22*a*) are both given as unity.

Experienced engineers will refuse to accept the contention that the continuous beam may be cut over the central support without its load-carrying capacity being impaired. They may refer to their horse sense, or mention a conflict between theory and practice. Horse sense is of course not infallible, but in this instance it seems that it is superior to the elastic-strength theory.

We shall offer a theoretical analysis of the functioning of both the simple and the continuous beam under continuously increasing loads. This theoretical analysis is backed by experimental test results as shown on pages 49–65.

Let

$$\text{The beams be W.F. } 10'' \times 10'' \times 49 \text{ lb}$$

$$\text{The moment of inertia, } I = 273 \text{ in.}^4$$

$$\text{The section modulus, } I/c = 54.6 \text{ in.}^3$$

$$\text{The elastic-limit stress, } s_1 = 30{,}000 \text{ psi}$$

$$\text{The modulus of elasticity, } E = 30{,}000{,}000 \text{ psi}$$

The area A in formula II is

$$A = 7.2$$

The distance $\bar{y}$ in formula II is

$$\bar{y} = 4.125$$

Therefore, $A\bar{y} = 29.7$, and, from formula II, the limit resisting moment becomes $M = 2s_1A\bar{y}_1 = 59.4s_1$.

As the beam (Fig. 21) is loaded beyond the range of elasticity, it develops a ductile flow, the limiting outline of which is indicated by the black spot in the center of Fig. 21*b*. Figure 21*f* represents the stress distribution at section A. Figure 21*g* shows stresses at section B, while Fig. 21*h* shows the stresses at section C of Fig. 21*b*.

Two shear curves (Fig. 21*c*) and two bending-moment curves (Fig. 21*d*) are shown, one for the load w_1 and the other for the load w_2. The deflection $\Delta_1 = 0.30$ in. (Fig. 21*e*) corresponds to the loading w_1 when the elastic limit in the beam has just been reached.

Figure 22a represents a W.F. 10″ × 10″ × 49 lb per ft beam (the same size beam as that shown in Fig. 21a) spanning two equal openings of 10′–0″ and loaded with a uniformly distributed load.

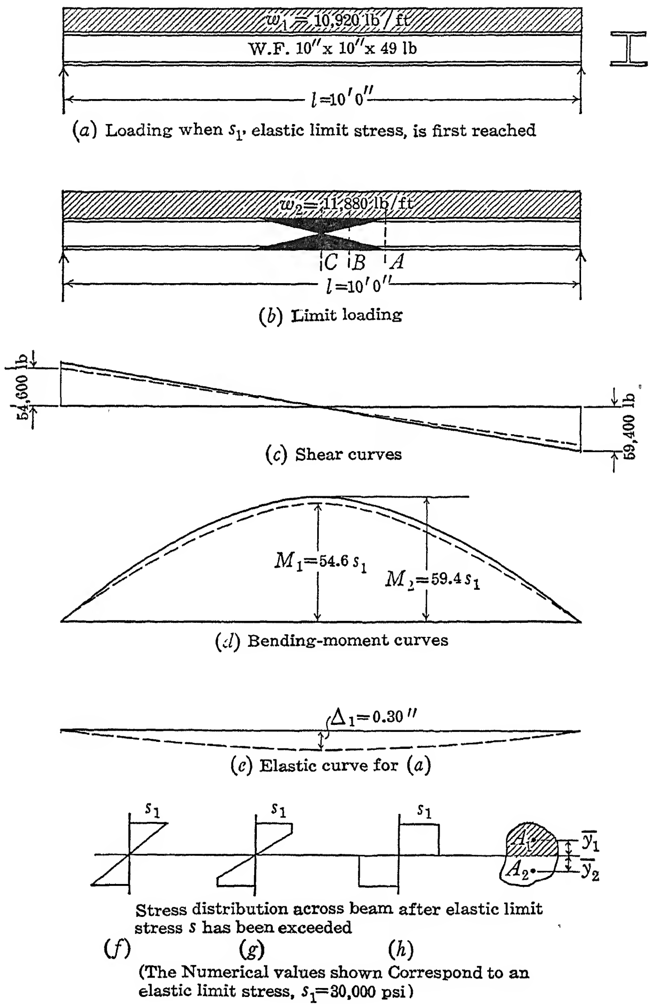


FIG. 21.

Its bending-moment curve, as long as all the stresses are strictly within the elastic limit, is represented by the flattest of the three curves shown in Fig. 22e. Note that the maximum bending moment occurs over the center support and equals $0.125w_3l^2$.

ıerefore, $w_3 = w_1 = 10{,}920$ lb per ft is the uniformly distrib- load which would bring the stress in the extreme fiber over support up to the elastic-limit stress, $s_1 = 30{,}000$ psi. The ction Δ_3 (Fig. 22*f*) for this loading is 0.126 in.

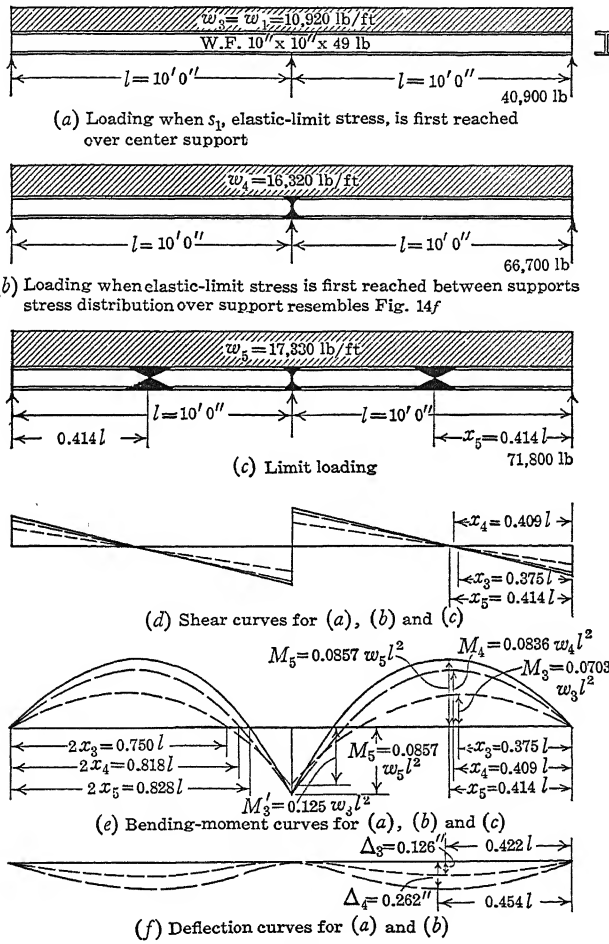


FIG. 22.

If next we increase the load on the beam beyond w_3, ductile flow over the center support takes place. The bending moment over the center support increases slowly, in agreement with the stress pattern shown in Fig. 21*g*. The ductile flow soon extends nearly all the way down to the center of the beam, and the stress

distribution over the support is then nearly as that shown in Fig. 21*h*. By this time, as we have seen in the previous example, the value of the resisting moment over the support is substantially as given in formula II, and is only 8.8 per cent greater than it was when the elastic limit was first reached. This resisting moment is

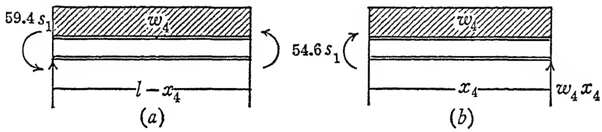


FIG. 23.

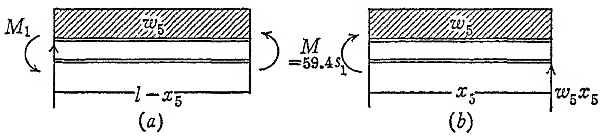


FIG. 24.

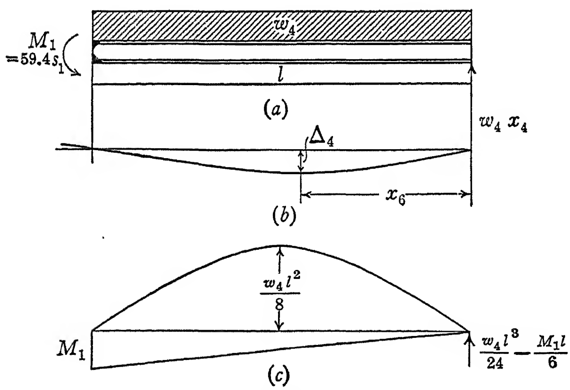


FIG. 25.

the maximum moment that can be developed over the center support. While the bending moment over the support is slowly increasing, or remains substantially constant after its maximum value has been reached, the maximum bending moment between supports increases rapidly with a continued increase in loading, and the position of maximum bending moment shifts closer to the center support. A condition will result in which the elastic-limit stress in the outer fiber between supports is reached. Upon additional loading, ductile flow between supports takes place and continues until the stress pattern (Fig. 21*h*) prevails. When that

condition is reached (Fig. 22c), the deformation progresses unhindered. The beam distorts, twists out of shape, collapses. The load w_5 which causes this condition is the limit load. Under any lesser load, even a small percentage less than w_5, the ductile flow, indicated by the black triangles (Fig. 22c), does not extend entirely throughout the beam. There is thus present a completely elastic core between supports. Deformations, therefore, are still of the order of magnitude of elastic deformations and may be computed with the same degree of accuracy with which we are accustomed to compute elastic deformations (see Fig. 16 and E. O. Scott's dissertation, reference 5).

We compute the value of w_4 and w_5 (Fig. 22) in the following manner: We define the distance from the right support to the point of maximum moment between supports as x_4 in Fig. 22b and x_5 in Fig. 22c. We proceed to separate the beam over the right span into two parts by passing a plane through the point defined by either x_4 or x_5, and also by passing planes through the outer and center reactions. We then draw free body sketches of these parts (Fig. 23 and Fig. 24). The statement that the shear is zero at the point of maximum bending moment expresses an equilibrium condition (statics) and is equally as valid after ductile flow has taken place as it was before. Therefore, at the points in the beam marked by x_4 and x_5, the shear is zero. From Fig. 23 we may write

$$\frac{w_4 x_4^2}{2} = \frac{I}{c} s_1 = 54.6 s_1$$

$$\frac{w_4(l - x_4)^2}{2} = 54.6 s_1 + 59.4 s_1 = 114 s_1$$

Solving these equations, we obtain

$$w_4 = 16{,}320 \text{ lb per ft} \qquad \text{and} \qquad x_4 = 0.409l$$

From Fig. 24 we may write

$$\frac{w_5 x_5^2}{2} = M_1$$

$$\frac{w_5(l - x_5)^2}{2} = 2M_1$$

Solving these equations we obtain

$$w_5 = 17{,}330 \text{ lb per ft} \qquad \text{and} \qquad x_5 = 0.414l$$

The problem of finding the deflection Δ_4 for the beam, shown in Fig. 22*b*, resolves itself into finding the maximum deflection for a simple beam having a predetermined moment, $M = 2s_1A\bar{y} = 59.4s_1$, at its slightly rounded left end (Fig. 25*a*). The distance to the point where the maximum deflection occurs, thus computed, is found to be $x_6 = 0.454l$ and the corresponding deflection, $\Delta_4 = 0.262$ in.

In comparing the relative merits of the beams and their various loadings, shown in Figs. 21 and 22, we are forced again to consider critically what constitutes the true criterion of strength of steel beams. In this connection we call attention to the fact that in 1912 (reference 2) P. W. Bridgman published some experiments which challenge the concept stress as criterion of strength. By placing metal and glass rods in a box, allowing the ends to protrude, and subjecting the parts of the rods within the box to hydrostatic pressure, he succeeded in fracturing the rods along planes of zero stress. It would seem that in building construction any beam may be said to function properly so long as the deformations are small enough to prevent the plaster underneath from cracking. In railroad-bridge stringers we do not want any stringer to deflect 20, 30, or 50 times as much as the one placed in a symmetrical position to it. We have seen (Fig. 22*b*) that the load may be increased 50 per cent over w_3 to equal $w_4 = 16{,}320$ lb per ft, with a corresponding deflection $\Delta_4 = 0.262$ in., a deflection smaller than the elastic deflection $\Delta_1 = 0.30$ in. given in Fig. 21*e*. In order to carry the increment of load beyond the 10,920 lb per ft, the beam must suffer a ductile yield at the point over the center support. Traditionally, we are supposed never to countenance exceeding the elastic limit. But why frown on it, *a priori*, when we continually exceed the elastic-limit stress as we hammer out dents and cold-straighten beams before fabrication.

The reaching of the elastic-limit stress in the center of beam (Fig. 21*a*) serves as a criterion of strength, because it marks the loading, which, if exceeded by only a small percentage, will result in disastrous distortions, or collapse. The reaching of the elastic-limit stress over the center support (Fig. 22*a*) merely means that a readjustment of stresses will commence. Disastrous distortions, or collapse, in the beam (Fig. 22) will not begin until the load w_3 has been exceeded by 50 per cent, and the load w_4 has been reached. After the load w_4 is reached, we have only a margin of 6.2 per cent

increase in load before load w_5 is reached, which loading induces unrestricted distortions or complete collapse.

Design Procedure. In 1917 Professor N. C. Kist (references *3a* and *3b*) formulated the following rule:

In the design of a redundant structure, it is not necessary to use the equations of elasticity to determine the redundants; it is only necessary to assume values for them, any assumptions at all but preferably the most advantageous ones, provided such assumptions are compatible with the conditions of equilibrium.

In our analysis of the continuous beam of two equal spans, over three supports, and loaded with a uniformly distributed load, we found that, so long as the beam functions elastically, the ratio of the two critical bending moments, the one over the central support and the one between supports, is $0.125/0.0703 = 1.78$. We have also shown that, if the loading continues until the moment over the support is that given by formula II, $M = 2s_1A\bar{y}$, and the moment between supports equals $M = s_1I/c$, then the maximum deflection equals $\Delta_4 = 0.262$ in. This deflection is 14 per cent less than the deflection of the simple beam (Fig. 21*a*) when the elastic limit is just reached. Since this deflection, $\Delta_1 = 0.30$ in., of the beam shown in Fig. 21*a*, is never questioned, it would seem unreasonable to question the smaller deflection, $\Delta_4 = 0.262$ in., as a permissible one except under certain circumstances, such as reversal of loading, or thousands of repetitions of loading. The procedure which we followed by basing our strength analysis of the continuous beam over three supports on two critical moments, one given by the formula $M = 2s_1A\bar{y}$ and the other by $M = sI/c$, seems justified in our academic discussion but seems also an unwarranted refinement for conventional design procedure. It is suggested that in a conventional design procedure the two critical bending moments be considered equal, with the understanding that, if we consider these two bending moments as $M = sI/c$, we err slightly on the safe side; and if we consider both these moments as $M = 2s_1A\bar{y}$, then we err slightly on the unsafe side. Given a choice between the two, we prefer to have our error slightly on the safe side, and thus recommend the use of the elastic-stress formula, $M = sI/c$. This has an added advantage in that the term I/c is readily available, whereas the term $A\bar{y}_1$ is as yet not to be found in any handbook.

A graphical analysis much simpler than the one given previously is here suggested. The elastic bending moment, shown by the

flattest of the three curves (Fig. 22*e*), may also be drawn as shown in Fig. 25*c* or in Fig. 26. The dashed base line (Fig. 26) is the one that prevails so long as the beam behaves elastically, while the solid base line prevails under limit loading conditions. If the maximum rise of the parabola represents $wl^2/8$, then the solid base line is so constructed as to give equal ordinates M_5 over the central support and between the central and the end supports. In our case these equal ordinates are $0.0857w_5l^2$, as appears in the left half of Fig. 26. If we make our analysis graphically, we may not be able to obtain an accuracy of three significant figures, but we will have an accuracy sufficiently close for design.

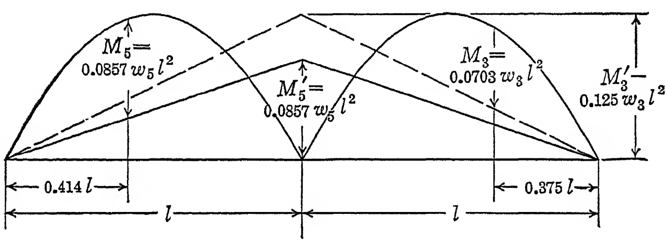


FIG. 26.

Once having determined the maximum moment, $M = 0.0857w_5l^2$, we proceed in the conventional manner of design. We evaluate $I/c = 0.0857w_5l^2/s_1$ in which I/c represents the section modulus of a beam that will just fail under a limit load w_5, or we say that $I/c = 0.0857wl^2/s$ in which s represents working stress, w represents safe loading, and I/c represents the section modulus of a beam that will safely carry load w. The two procedures evidently give identical answers. In the analysis of the built-in beam, following the same rule, the logic of limit design indicates that the X axis be placed as shown by the dashed line in Fig. 20*d*, dividing the ordinate $wl^2/8$ in two equal parts.

As one more example of a design (by means of the theory of limit design) of beams which are to carry loads in fixed positions, we offer the analysis illustrated by Fig. 27. First we draw the bending moments to scale, assuming the structure to consist of a number of simply supported beams. Next we fit a base line to these bending-moment curves in a manner so as to make the maximum ordinates most favorable (Figs. 27*b* and 27*d*). Suppose we decide to make the beam continuous and of constant cross section throughout. We observe then that anything we might do with

the base line between points A and E would have no effect on the base line between points E and F. The maximum moment between points E and F scales 593 in. kips.

In the interest of economy we may cut the beam at D, splice it at B' (Fig. 27c), and design the portions AB' and $B'D$ for their limit moments, 2×432 in. kips, and 2×263 in. kips, respec-

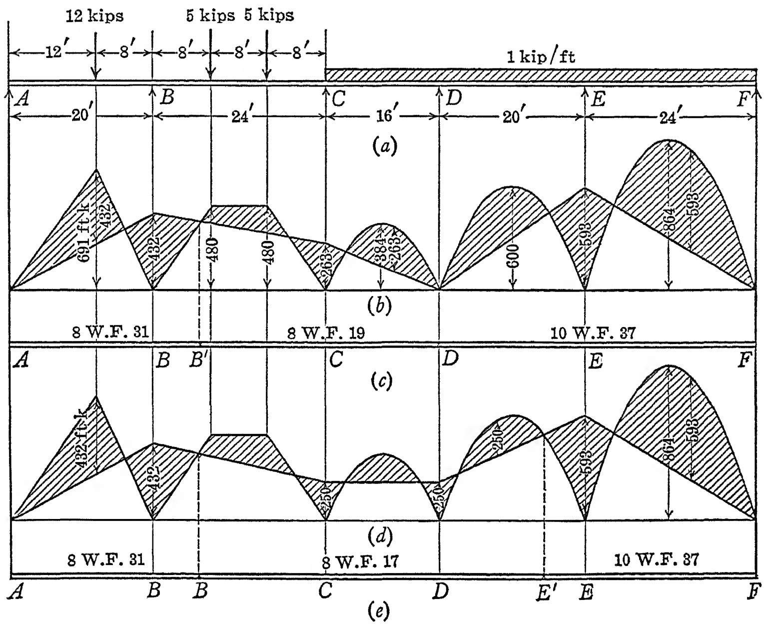


FIG. 27.

tively, which values are scaled from Fig. 27b, the factor 2 being an overload factor.

If instead of cutting the beam over the support at D we splice it at E' (Fig. 27e), then the bending-moment diagram may be drawn as shown in Fig. 27d. The beam between points B' and E' may then be designed for a limit moment of 2×250 in. kips.

In this problem we are not considering the advisability, or the feasibility, of splicing beams between supports. Its purpose is merely to illustrate the method.

The loads, inevitably, will have to be placed in their proper positions. This means that, though the loads shown in Fig. 27 may be the maximum loads, we must consider the order in which the loads are applied.

Alternate Loading. The limit bending moment for the limit load w_5, shown in Fig. 22*c*, is reproduced again in Fig. 28*a*. Removing the limit load w_5 (Fig. 28*a*) is equivalent to superimposing on

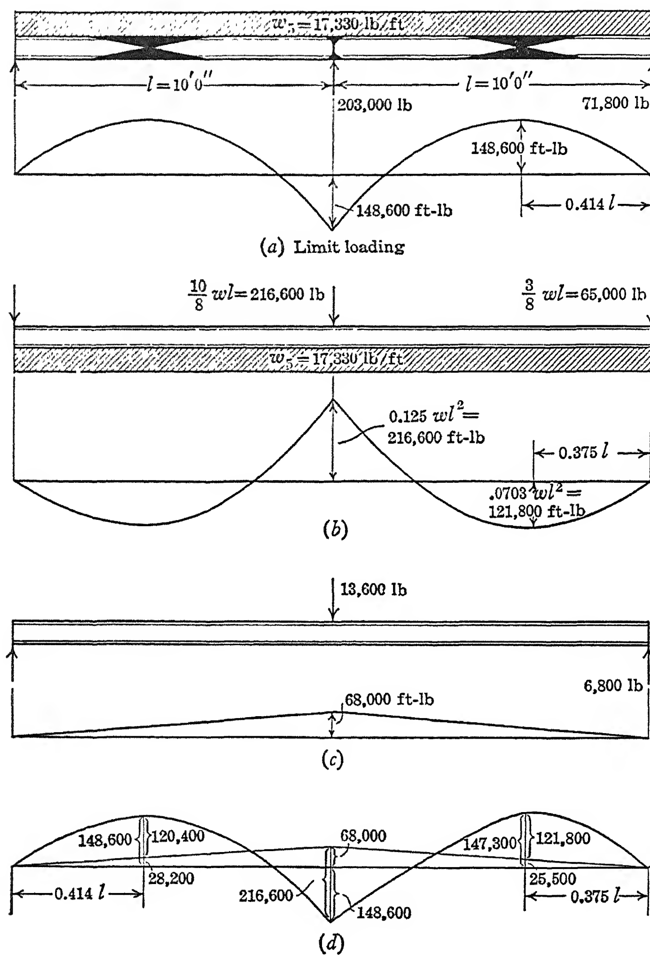


Fig. 28.

this load an equal loading of negative sign. When the stresses in steel are first reduced and then reversed, after they have previously exceeded the elastic limit, the steel behaves in a manner as exemplified by curve CC' (Fig. 1); that is, it behaves elastically. This is an approximate assumption as was discussed previously

(page 27). For the negative loading w_5 the reactions and bending-moment curves appear, therefore, as shown in Fig. 28*b*. Superimposing these reactions and bending moments of Fig. 28*b* on those of Fig. 28*a*, we obtain the resultant residual reactions and bending moments as represented by Fig. 28*c*. We have assumed the possibility of the reaction at the middle support changing sign. If the beam merely rests on single knife-edge supports, that is, if there is no restraint against upward vertical motion over the middle support, then it is impossible for the residual loading shown in Fig. 28*c* to develop. The beam will instead assume a curved shape in agreement with the moment diagram of Fig. 28*c*; that is, if the ends of the beam remain in contact with their support, then the center of the beam will rise from its support. This is what happens in a laboratory test when a continuous beam is strained beyond the elastic limit and the loading is subsequently removed.

If next the elastic loading and bending moments of Fig. 28*b* are reversed and superimposed on the residual loading and moment of Fig. 28*c*, then we obtain the resultant bending moment given in Fig. 28*d*, and the resultant loading and reactions of Fig. 28*a*.

It is significant that the limit loading and limit bending moments are now developed without any more ductile yielding being involved. As was formulated on page 26, once a limit load is applied and the necessary ductile yielding takes place, then, on removal of this load, residual stresses of the proper sign and magnitude remain in the structure. These residual stresses insure the perfect elastic functioning of the structure on subsequent application of the limit load, provided this loading is applied in the same sense.

If a load, say w_4 (Fig. 22), be applied [a load greater than the original elastic load w_3 (Fig. 22) but smaller than the limit load w_5], then residual stresses of the correct magnitude and sign will be induced. Thus residual stresses insure perfect elastic functioning, without ductile yielding, under application of any loading less than w_4.

If a load of reverse sign be applied, say a loading acting upward instead of downward, it appears that the residual loading and bending moments are of a sign such that the elastic limit will be reached sooner than it would if no residual stresses were present. In this instance, as in that discussed on page 25, when loading is reversed, the sum of the positive and negative loading is not to

exceed twice the elastic loading, if accumulative ductile yielding is to be avoided. When positive and negative loads are of the same magnitude, we are restricted to purely elastic loading.

The bending-moment curves shown in Fig. 28*d* and Fig. 26 are identical except for a shifting of base line. Figure 29 represents the same beam shown in Fig. 22 and Fig. 28, but here loaded with a uniformly distributed load over only one span. If we follow the rules established in the previous pages and compute the limit load, we find it to be equal to that found when both spans are fully loaded with a uniformly distributed load. It is significant to observe, however, that under elastic loading (Fig. 29*b*) the greater bending moment occurs between supports. This is the point, then, where ductile yielding will first take place. Note in contrast that under a uniformly distributed load (say w_4 of Fig. 22*b*), ductile yielding first commenced over the center support.

Figure 29*a* represents the limit loading and limit bending moments.

Figure 29*b* represents the loading and bending moments under a load equal to the limit loading, but of opposite sense, while the beam behaves elastically.

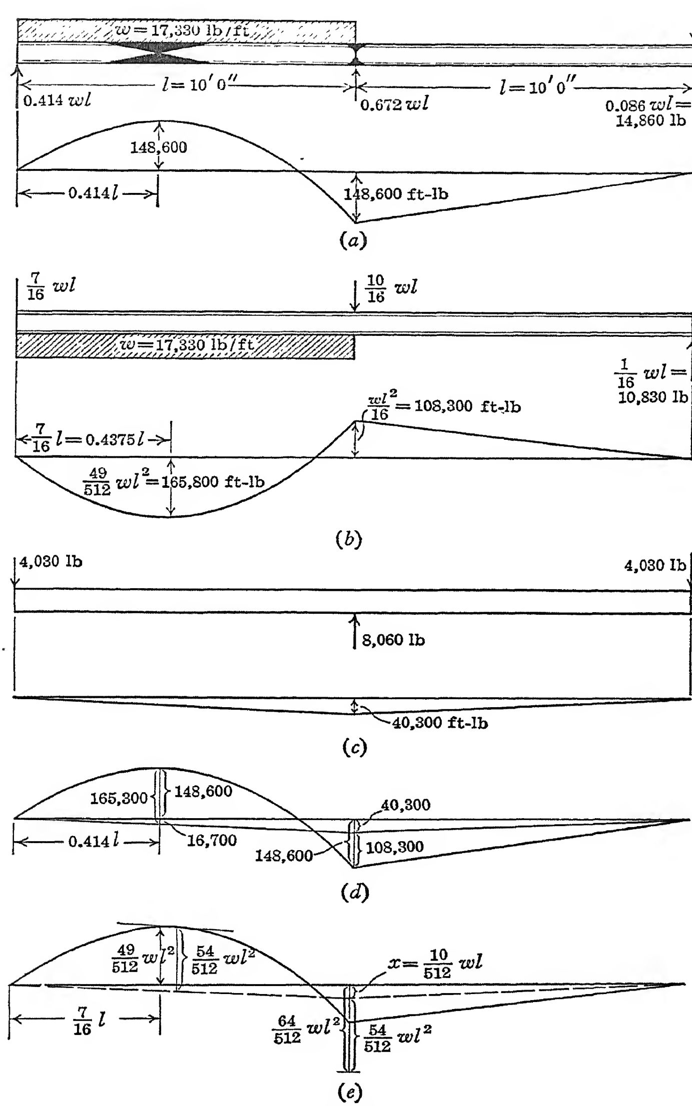
Figure 29*c* represents the residual loading and bending moments resulting from the removal of the limit load shown in Fig. 29*a*.

Figure 29*d* shows how the same limit bending moment of Fig. 29*a* is obtained in a purely elastic manner when the limit loading is applied to the beam after the residual reactions and bending moments (Fig. 29*a*) have previously been induced.

Figure 29 shows nothing that was not already offered and discussed in connection with Fig. 28; and the discussion relative to Fig. 28 is largely a repetition of the analysis of Fig. 17. The object in offering Fig. 28 and Fig. 29 is to have them as references in the discussion of alternate loading.

It may be observed that, after the limit loading has been applied and removed, the residual reactions and bending moments (Figs. 28*c* and 29*c*) not only are of different magnitude but also differ in sign.

Assume the limit loading w_5, extending fully over both spans (Fig. 28*a*), to be applied and removed. This procedure, as we have seen, leaves the beam stressed as shown by Fig. 28*c*. If next we load the beam with a uniformly distributed load, extending over only one of the spans (Fig. 29), then, the residual bending moment being of unfavorable sign, the elastic-limit stress will be reached



w = 17,330 lb/ft
l = 10' 0"
l = 10' 0"
0.414 wl
0.672 wl
0.086 wl = 14,860 lb
148,600
0.414l
148,600 ft-lb
(a)
7/16 wl
10/16 wl
w = 17,330 lb/ft
1/16 wl = 10,830 lb
wl²/16 = 108,300 ft-lb
7/16 l = 0.4375 l
49/512 wl² = 165,800 ft-lb
(b)
4,030 lb
4,030 lb
8,060 lb
40,300 ft-lb
(c)
165,300
148,600
40,300
0.414 l
16,700
148,600
108,300
(d)
49/512 wl²
54/512 wl²
x = 10/512 wl
7/16 l
64/512 wl²
54/512 wl²
(e)

FIG. 29.

earlier than under zero residual stresses. This uniformly distributed load extending over only one span may be increased to the value of the limit load w_5, but this can be done only at the expense of additional ductile yielding. When this limit load is removed, the residual-stress pattern corresponds to that in Fig. 29*c*. If subsequently a uniformly distributed load extending over both spans is again applied, it is found that this residual-stress pattern is of unfavorable sign. The elastic limit will again be reached sooner than under zero residual stresses. The load can again be increased to the value of the limit load w_5, but not without additional ductile yielding. A uniformly distributed load of a magnitude of w_5, applied alternately first over one and then over both spans, induces ductile flow under each application of the load. This would ultimately, and probably very soon, result in failure.

Although the load w_5, when thus alternately applied first over one and then over both spans, is too high to be regarded as a limit load, this does not mean that some other limit load, say w_6, may not be applied in this alternate manner indefinitely.

To evaluate the limit load w_6, which may thus alternately be applied an indefinite number of times without causing failure, let us begin the value of the load at zero and increase it by very small increments after each alternating application. Under elastic loading, the maximum moment in Fig. 28 is $wl^2/8$ or $^{64}/_{512}wl^2$, and it occurs over the center support, whereas in Fig. 29 it is $^{49}/_{512}wl^2$ and occurs between supports. Therefore, as the load w, alternately applied in the manner of Fig. 28 and Fig. 29, increases, the elastic limit over the center support is reached when w has a value of w_3 (Fig. 22*a*). As soon as w_3 is exceeded, a residual moment will be built up. This residual moment is smaller in magnitude than the residual moment shown in Fig. 28*c* but is qualitatively identical with it.

The question now is to find w_6, the maximum value of the load alternately applied in the manner of Figs. 28 and 29, which will build up a residual-stress pattern favorable to the application of a load in the manner of Fig. 28, and yet not so large as to become unfavorable to the application of a load in the manner of Fig. 29. How this may be done is illustrated in Fig. 29*e*.

The solid black line, Fig. 29*e*, shows the maximum moment when the load is applied in the manner of Fig. 29. The maximum moment $wl^2/8$ or $^{64}/_{512}wl^2$ is laid off at the position of the middle support. This maximum moment occurs when the load is applied

in the manner of Fig. 28. The dash line shows the residual moment which is of a magnitude such that, when, between supports, its ordinate is added to the ordinate of the bending moment induced by a loading in the manner of Fig. 29, and, over the support, its ordinate is subtracted from the bending moment induced by a loading in the manner of Fig. 28, the maximum ordinates in the two cases are equal. This maximum ordinate between supports occurs at a point where the tangent to the solid-line curve is parallel

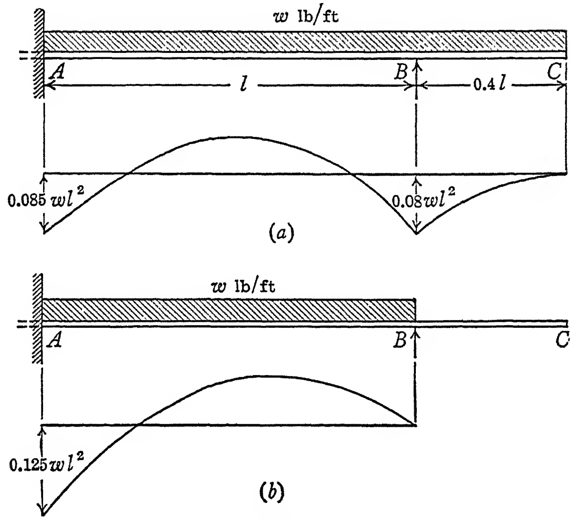


FIG. 30.

to the dash line. This point may be found analytically. In our case it is obtained graphically with sufficient accuracy. This maximum moment scales $\frac{54}{512}wl^2$. The w in this last expression is the limit load which, when alternately applied in the manner of Figs. 28 and 29, may be applied indefinitely without inducing accumulative ductile yielding. This then is the limit load w which we set out to find.

By formula II,

$$\tfrac{54}{512}w_6l^2 = 2s_1A\bar{y} = 59.4 \times 30{,}000$$
$$w_6 = 14{,}910 \text{ lb per ft}$$

For a more detailed discussion of this problem see reference 6. For a discussion of moving loads see reference 1b.

Figure 30a represents a beam fully loaded with a uniformly distributed load. It also shows the maximum bending moments

under purely elastic behavior. Figure 30*b* represents the same beam partially loaded, and its corresponding elastic bending moment. It appears that if the beam is designed by the criteria of the theory of elasticity the largest moment occurs under partial loading (Fig. 30*b*). The controlling moment for which the beam should be designed is $0.125wl^2$.

If the same beam were designed by the rules of the theory of limit design, the moment $0.08wl^2$ over the support at B when the beam is fully loaded, would be the controlling moment. Alternate loading in this case would not affect this value.

Factor of Safety. It may be argued that we have taken liberties with the factor of safety. This is not our intention. No one alone can decide on the proper factor of safety. We have merely tried to illustrate a method. This method is as independent of the assumed factor of safety as it is independent of the elastic-limit stress, also assumed. If we insist on applying conventional methods and theories, then, in order to take advantage of continuity, we should specify one working stress for the simple beam and another working stress for the continuous beam. This means in effect that we should specify a variable factor of safety for different structures built of the same structural steel. The theory of limit design offers a means whereby a constant factor of safety may be selected. This factor would come closer to providing a measure for the margin of safety in our structure than does the one now applied and by means of which we decide working stresses.

The following is a quotation from a discussion of reference 1*b* submitted by Dr. L. H. Donnell, research professor of mechanics, Illinois Institute of Technology, and formerly in charge of stress analysis and structural research, Goodyear–Zeppelin Corporation.

The factors of safety used in many branches of engineering are based at least in part on practical experience of the past in similar work. If they were too low in the beginning too many failures occur, forcing engineers to raise them. Or if they were higher than necessary, in time some daring pioneers are sure to try lowering them, and if their experiments are successful other engineers eventually copy them. Thus values are gradually evolved which at least on the average are reasonably close to what they should be.

But obviously this process is valid only when the same methods of analysis are used on the new and old designs. If we do not improve the type of design, and introduce a new method which gives more optimistic estimates of the strengths of our structures, factors of safety based on past experience should

presumably be raised enough so that structures designed by the new method will on the average be as strong as those designed by the old methods.

This does not at all mean that the new method would give the same results in individual cases as the old, and hence that it has no advantage other than its greater simplicity. On the contrary, a design method which gives an imperfect measure of the *real* useful strength of a structure, as elastic methods evidently may, can obviously result in too low a real strength in some cases and too high a real strength in others, even if it is regulated to give correct results on the average. And conversely, any method, such as the one under discussion, which gives a better measure of the real useful strength, will always give results closer to what they should be in individual cases.

Experimental Results. The built-in beam, discussed earlier, is a typical textbook problem. It is fictitious, as a perfect built-in condition is unattainable either in practice or in the laboratory. Uniformly distributed loading is also very difficult to realize in the laboratory. The equivalent of the ideal built-in condition, however, can be obtained by taking advantage of symmetry, and the theory may be checked for concentrated loads as well as for uniformly distributed loads. The following record is taken from the appendix of reference 1*k*. See also reference 7.

Figure 31*a* represents a simply supported 3-in. steel I beam, with a moment of inertia equal to 2.9 in., loaded at the third points. Figure 31*b* represents an identical 3-in. steel continuous I beam loaded and supported as shown. Let P_1, P_2, P_3, P_4, and P_5 represent successively increasing loads. Let 1, 2, 3, 4, and 5, of Fig. 31*c* represent successively increasing bending moments, corresponding to the loads P. Figure 31*d* represents successive stress distributions over the cross section of the beam at B. Figure 31*e* represents successive stress distributions over the cross section at point C. Figure 32 shows the load-deflection curves for both the simple and the continuous beam. The theory of elasticity tells us that, so long as the beam functions elastically, the ratio of the bending moments at points C and B in Fig. 31*b* is $M_c/M_b = 1.875$. The ordinates to the bending moments at points C and B for bending moments 1, 2, and 3, as well as the maximum stresses, indicated by 1, 2, and 3 in Figs. 31*e* and 31*d*, are thus in this same ratio. As load P_3 is increased until it reaches the value of P_4, the resisting moment at point C is increased 12 per cent while the stress distribution at section B is represented by no. 4 on Fig. 31*d*. If load P_4 is further increased to P_5, then the maximum resisting moment at point C remains constant and the stress distribution over the cross section at B changes from that represented by no. 4

to that represented by no. 5 (Fig. 31d). The moment at B increases simultaneously some 12 per cent.

For the loads P_1, P_2, and P_3, within the elastic range, the bending moments 1, 2, and 3 of Fig. 31c all manifest the same point

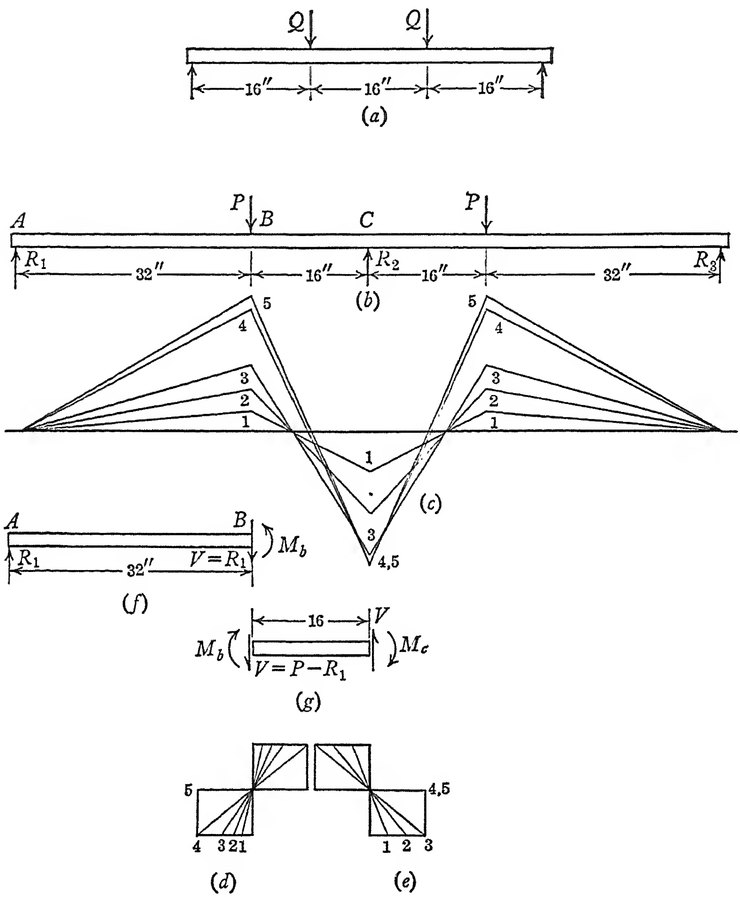


FIG. 31.

of inflection. As P_3 is increased to P_4, the bending moment at C increases but slightly while the bending moment at B increases materially. The point of inflection thus moves nearer the central support. As P_4 increases to P_5, the bending moment at the support remains constant while that at point B increases some 12 per cent. Therefore, the point of inflection moves still farther

towards the central support, in fact, moves to the midpoint between points B and C.

The beams' behavior, outlined in the foregoing, may be traced on the load-deflection curves shown in Fig. 32. The elastic-limit load, $2Q$, on the simple beam is 7700 lb. The corresponding bending moment is $3850 \times 16 = 61{,}700$ in.-lb, and the corresponding elastic-limit stress is $s = Mc/I = 61{,}700/2.9 \times 1.5 = 32{,}000$ psi.

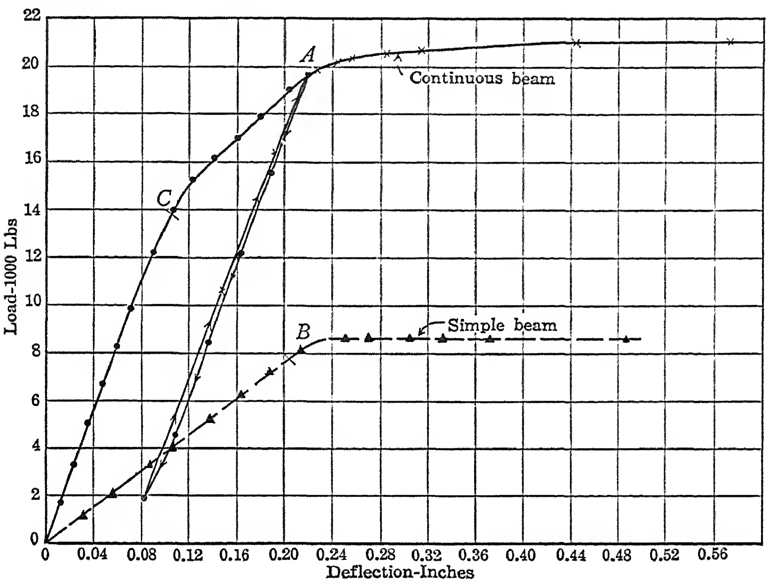


FIG. 32. Steel 3″ I Beams—Load-Deflection Curves of Simple and Continuous Beams. B and C mark elastic-limit loads; A marks limit load.

The limit load on the simple beam taken from Fig. 32, is 8600 lb or some 12 per cent greater than the elastic-limit load of 7700 lb.

In the analysis of the redundant beam the theory of elasticity gives M_c as a function of P, (Fig. 31b). $M_c = {}^{80}\!/_{9}P$ in.-lb. Taking the elastic-limit bending moment of 61,700 from the test record for Fig. 31a, and solving for P, we obtain

$$\tfrac{80}{9}P = 61{,}700 \qquad \text{or} \qquad P = 6900 \text{ lb}$$

In Fig. 32 the total maximum load within the elastic-limit stress, or $2P = 13{,}800$ lb, is shown as an ordinate and is marked C. Under a limit loading of the continuous beam $M_b = sI/c = 61{,}700$ in.-lb, and M_c is 12 per cent in excess of M_b. Thus $M_c = 69{,}000$.

Figures 31f and 31g represent free-body sketches of portions AB and BC of the continuous beam shown in Fig. 31b. Writing equilibrium equations for Figs. 31f and 31g, we obtain

$$32R_1 = 61{,}700 \qquad \text{or} \qquad R_1 = 1930$$

$$(P - R_1) \times 16 = M_b + M_c$$

$$16P - 16 \times 1930 = 61{,}700 + 69{,}000 = 130{,}700$$

Therefore, P = 10,100 lb, and $2P$, the limit load for the entire beam as recorded by the testing machine and as shown in Fig. 32, equals 20,200 lb. This limit load of 20,200 lb is 46 per cent in excess of the load of 13,800 lb which caused the elastic-limit stress at point C to be reached. The deflection of the redundant beam under this limit load of 20,200 lb is substantially the same as the elastic deflection of the simple beam.*

Aluminum and Magnesium Beams. Up to this point the theory of limit design has been predicated on the assumed elastic–ductile, stress–strain relationship as represented by curve $ABCC'$ (Fig. 1). Mild steel is a structural material the stress–strain properties of which closely resemble those indicated by Fig. 1. As illustrated by Figs. 16 and 32 the theory of limit design predicts and explains the behavior of structural-steel beams, after the elastic-limit stress is exceeded, quite satisfactorily. The stress–strain properties of alloys of the light metals, aluminum and magnesium, differ widely from those shown in Fig. 1. These metals are perfectly elastic up to a certain value of stress. Beyond this proportional-limit stress value the stress–strain curve is a smooth curve instead of a horizontal line (see Figs. 5, 6, 8, and 9). In the use of the light metals in airplane construction, the emphasis on weight-strength efficiency demands consideration of structural behavior when stresses exceed the proportional limit. The question arises whether the philosophy of limit design may, without extensive modification, be applied to the analysis and design of structures made of aluminum or magnesium alloys. A thesis by Dr. F. Panlilio deals with this problem (reference 8). His conclusions are as follows:

(1) The simplified methods of Limit Design, developed for specific application to steel structures, can be applied with safety to structures made of the alloys of magnesium and of aluminum.

* The test was performed under the author's direction by members of a class in theory of limit design. Figure 31 is taken from a report by Mr. John W. Luecht, and Fig. 32 from a report by Mr. Raymond Schneyer.

(2) The peculiar shape of the stress–strain curves of the light metals actually constitutes an advantage in that relatively higher loads can be carried by redundant structures of the light metals (as compared to non-redundant systems) than can be carried by redundant steel structures (as compared to non-redundant systems).

(3) The comparative brittleness of the light metals, however, constitutes a distinct disadvantage, which cannot be clearly reduced to exact figures until

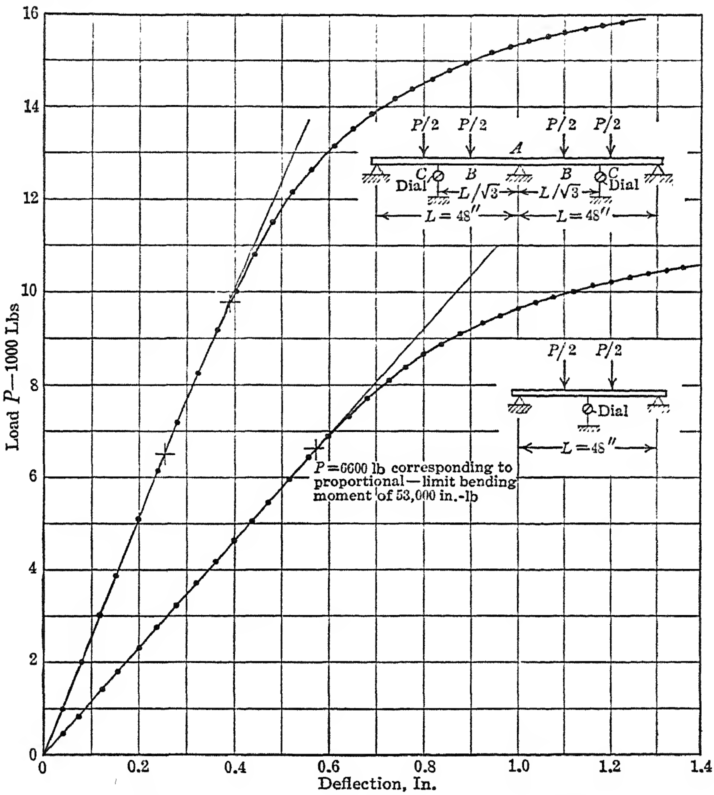


FIG. 33. R 61 ST Aluminum 1″ x 3″ Beams—Load-Deflection Curves of Simple and Continuous Beams.

an answer is found to the question: What is the minimum of ductility consistent with safety, considering the differences in the requirements of various construction?

(4) Thought should be given to adopting a criterion of strength more logical than the elastic limit stress. For the present there does not seem to be a more logical yardstick than the elastic deflection of non-redundant systems.

Dr. Panlilio's tests were largely confined to continuous beams on three supports, similar to the steel I-beam test which we have

just discussed (Figs. 31 and 32). However, instead of loading the continuous beam at one of the third points in each span, as was done in the test on the steel I-beam, he loaded the continuous beam with two equal loads placed at both third points of each of

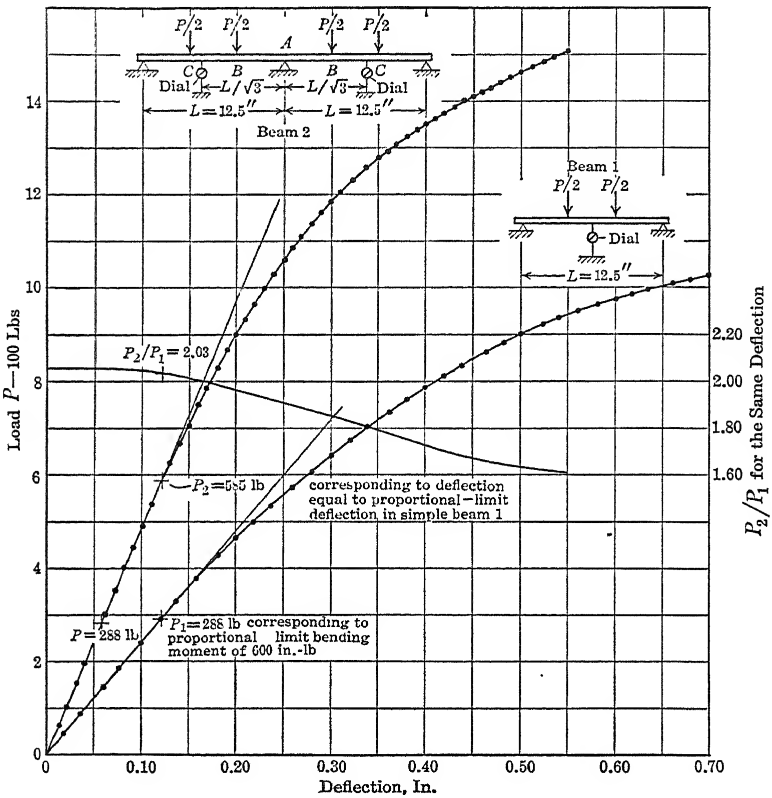


Fig. 34. ZK 60 Magnesium ⅝″ x ⅝″ Beams—Load-Deflection Curves of Simple and Continuous Beams.

the spans (Fig. 33). Under this type of loading the bending moment over the middle support (point A) of the continuous beam equals $Pl/6$, which is identical with the maximum bending moment in what he calls "the yardstick," namely, the simple beam subject to symmetrical concentrated loading at the third points. According to the elastic-stress criterion of strength, therefore, both simple and continuous beams should be equally strong. According to the permissible deflection criterion of strength (limit design) the continuous beam, for a deflection of 0.6 in., registers a strength

of almost 100 per cent in excess of that registered by the simple beam.

Figures 34 and 35 show similar test results for $\frac{5}{8}$-in. square

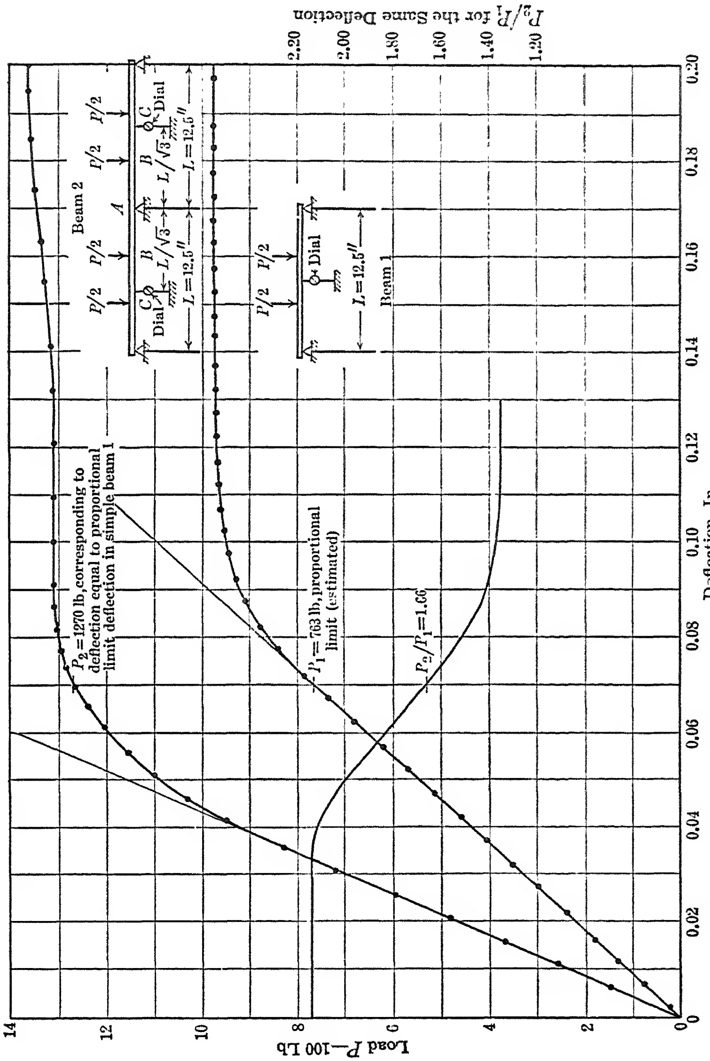


FIG. 35. Mild Steel $\frac{5}{8}''$ x $\frac{5}{8}''$ Beams—Load-Deflection Curves for Simple and Continuous Beams.

beams of ZK 60 experimental magnesium alloy and hot-rolled steel, respectively. Note that for the 61 ST aluminum and ZK 60 experimental magnesium alloy the ratio of the strength of the

continuous beam to that of the simple beam, corresponding to deflections equal to the proportional-limit deflection in the simple beam, is about 2, whereas for the hot-rolled steel beam it is 1.66.

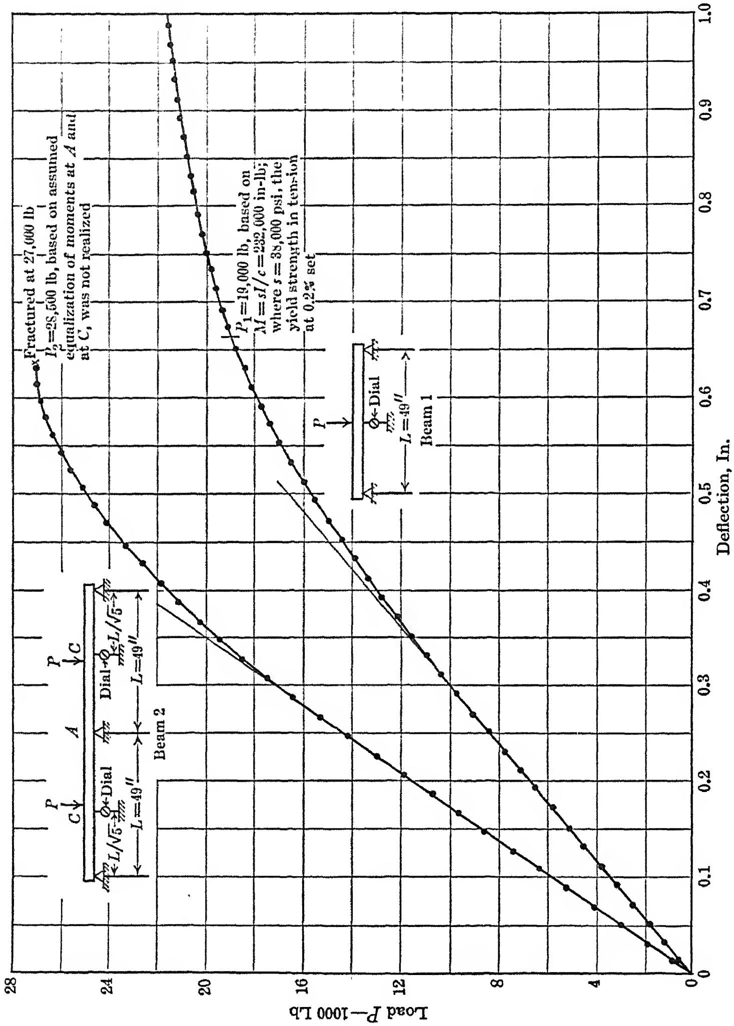


FIG. 36. 0-1 HTA Magnesium 5″ I Beams—Load-Deflection Curves for Simple and Continuous Beams.

Furthermore, in the aluminum and magnesium beams the load-deformation curves continue to rise, apparently indefinitely, whereas in the steel beam these load-deformation curves flatten out. This phenomenon led Dr. Panlilio to formulate statement 2

of his conclusion. Figure 36 shows the load-deflection curves for a simple and for a continuous 5-in. I-beam of 0-1 HTA magnesium alloy loaded as shown. The general shape of the curves is similar

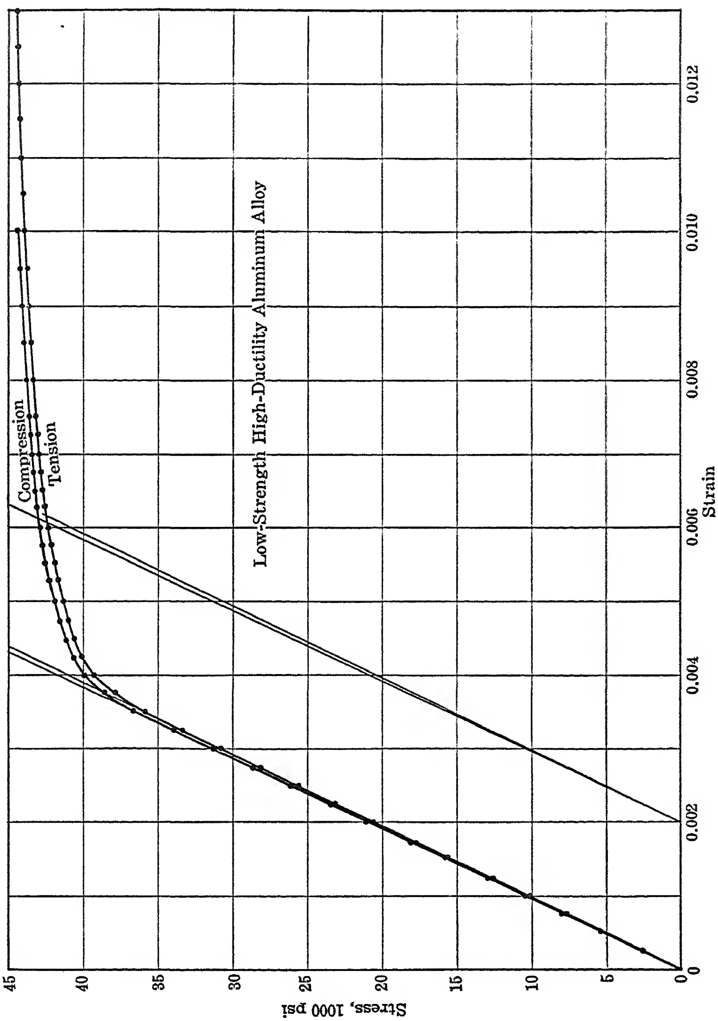


FIG. 37. R 61 ST Aluminum Extrusion Stress–Strain Curves.

to those of Figs. 33, 34, and 35. However, the full computed limit load was not reached, because the beam fractured apparently as the consequence of insufficient ductility. Figures 37 and 38 show tensile and compressive properties of 61 ST aluminum alloy and

ZK 60 experimental magnesium alloy. Figure 7 shows their relative ductile properties.

Two points should be emphasized in connection with the beam analysis we have just made.

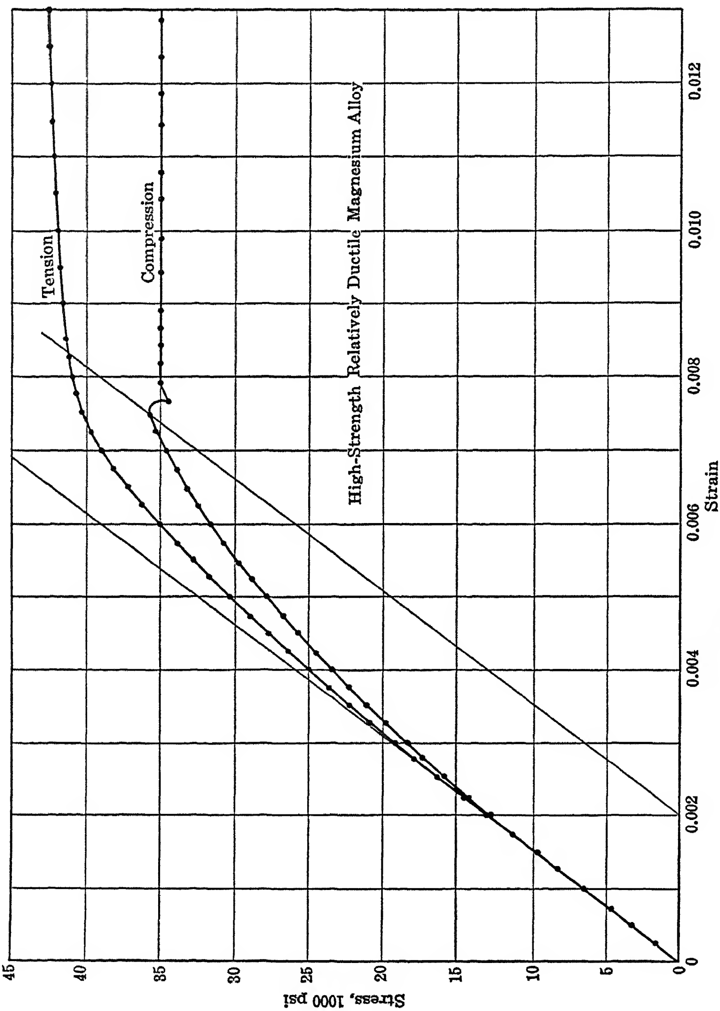


FIG. 38. ZK 60 Magnesium Extrusion Stress–Strain Curves.

We have directed our attention principally to the bending phenomenon. It is well recognized that short and stubby beams, when loaded to capacity, fail in shear or as the result of buckling of the web. The buckling strength is a function of the modulus,

and, since the modulus of aluminum and magnesium is one-third and one-fifth, respectively, that of steel, the condition of what may be called secondary web failure is reached in aluminum or magnesium beams much sooner than in steel beams. This consideration seems especially pertinent, because it appears that a majority of the magnesium beams listed in handbooks are copied after similar aluminum beams, and these aluminum beams in turn are copied after steel beams. Thus the required extra web thickness, necessary for beams to function as beams and not as columns over the supports, is not found in many of the commercial light-metal-alloy beams.

This point is further illustrated in structural handbooks with tables of "allowable uniform loads in kips." The simple beam, subject to uniformly distributed loading, is taken as a standard. Although uniformly distributed loading is rarely to be found in practice, these tables nevertheless serve effectively as a rough guide. In these tables are found horizontal lines both near the top and near the bottom. Those near the top mark the separation of the short beams from the longer ones, where longitudinal shear stress, or buckling strength of the web, is the deciding factor in strength rather than flexural stress determination of beams. Those near the bottom mark the separation of long beams from the shorter ones where the deflection may become the determining factor of usefulness.

Although predicated on simple beam behavior as a standard, the tables can as well be used for the design of built-in or continuous beams of equal span lengths. This is done by the simple process, in case the elastic-stress criterion of strength is followed, of multiplying all values (those for the very short beams excepted) by the factor $3/2$. We use the factor $3/2$ because the maximum bending moment is $wl^2/12$ instead of $wl^2/8$. But if limit-design criterion of strength is followed, we multiply all such values by a factor of 2. The exception "for very short beams," as noted in the foregoing sentence, is taken for the following reason: As the loads, because of continuous-beam action, are increased, the end shears are also increased. Thus, in effect, the horizontal line near the top (in the "allowable load on simple beam" table), marking the distinction between beams which will fail in shear from those which will fail in bending, needs to be dropped. If limit-design criterion of strength is followed, this drop is greater than it is when we follow the elastic-stress criterion of strength.

As shown in Figs. 37 and 38 the stress–strain curves of 61 ST aluminum and ZK 60 experimental magnesium alloy, respectively, exemplify the properties of two light-metal alloys, which to us represent the best now available for structural purposes. High strength, commonly emphasized by the stress analyst, may be misleading, as witness the tensile strength of stainless steel equal to 200,000 psi coupled with a compressive proportional limit stress of only 40,000 psi (see Fig. 11). Compressive properties of materials are commonly the controlling ones. These properties depreciate the value of stainless steel as a structural material. Furthermore, one of the most unfortunate misconceptions is to mistake strength of materials for theory of strength.

The new aluminum alloys, 75 ST and R 303 T, do not exhibit the undesirable properties of stainless steel. Their proportional limit and yield stress, both in tension and compression, are of the order of magnitude of 60,000 and 75,000 psi. These alloys, however, are expensive and relatively brittle.

The aluminum alloy 61 ST (in Canada 65 ST) has a proportional limit of 35,000 and a yield stress of 42,500 psi (Fig. 37). Cost figures are very difficult to obtain, but we believe it to be one of the cheapest structural aluminum alloys. Its welding characteristics are excellent, probably superior to all other structural aluminum alloys. On these scores alone it compares favorably with the alloy 24 ST, at present the most commonly used aluminum alloy in airplane construction. What recommends 61 ST aluminum alloy to us, however, is its markedly superior ductility (see Fig. 7) as compared with those of other light-metal alloys.

Of the magnesium alloys, 0-1 HTA, until recently, was considered best for structural purposes. Although we recognize its assets of strength and lightness (see Fig. 10), we judge it unfavorably because of its lack of ductility (see Fig. 7). Our Fig. 36 shows how the 0-1 HTA magnesium I-beam broke before the full-limit loading was reached. Because of our limited experience with magnesium and aluminum, and on the other hand, our extensive work of many years with steel, our judgment may well be somewhat prejudiced. The Dow Chemical Company has now developed an experimental magnesium alloy ZK 60, the stress–strain properties of which are shown in Fig. 38, the ductile properties in Fig. 7, and the load-deflection properties for beams in Fig. 34. But we have no knowledge as to the possible cost of the metal. Neither do we know whether or not it is available commercially. Ignoring the

cost and weight factors, we would hold it comparable to 24 ST aluminum alloy. Considering the weight factor, however, it seems definitely superior to 24 ST aluminum for airplane construction. Having just argued the superiority of 61 ST aluminum over 24 ST aluminum, we hold that the principal contestants for the award of merit as material for airplane construction are 61 ST aluminum and ZK 60 magnesium—if and when the latter metal becomes available in commercial quantities.

Chapter 4

LIMIT DESIGN OF TRUSSES

Objectivity is a desideratum, like an ideal, never humanly attainable. The writing of a text has one serious drawback, in that the views expressed may, unwittingly, take on an air of finality and remain publicly unchallenged. Throughout this text I have repeatedly stressed sense of value. About all I have had to offer has been my own "sense of value." The best I have been able to do is to present such support from experimental evidence and such support from logic as I have been able to muster. I will claim but one thing, not that I have been right but that I have been honest. After accepting or rejecting the offered experimental evidence and logic, the reader is left to draw his own conclusions.

The attempt to apply the philosophy of limit design to trusses involves the behavior of columns. In one of its first public statements, the recently formed Organizing Committee of Column Research Council, March 21, 1945, expressed itself as follows:

> The field of designing and detailing compression members in engineering structures is a confused one, and is becoming more so as new materials are produced and proposed for structural use. It is well known that specification-writing bodies prescribe different column formulas to meet the same condition, and that others prescribe the same formula to cover quite different conditions.
>
> It is not as generally understood as it should be that there is no possibility of applying a single column formula to all kinds of uses of columns, unless it contains several other factors besides length-ratio, and thus becomes most unwelcome in its complexity.

In 1941 I published a paper entitled "Rational Column Analysis" (reference 1*e*) in which I formulated what seemed to me certain elementary, basic, simple, and self-evident propositions. These views were strongly opposed by some of the discussers (reference 1*f*). When mature, honest, and competent men hold to basically opposite views on the column question, it seems to me that the Column Research Council selected a mild adjective when it referred to this state of affairs as "confused."

Before starting to state my views on columns, I am thus aware that these views may be disagreed with, objected to, and even resented. However, I see no clear way out of this dilemma. The reader inevitably will have to take sides. I do not propose to discuss the entire column phenomenon. I shall limit myself to such aspects as have a bearing on limit-design philosophy. To refer again to "sense of value," I shall try to emphasize the basic, the primary, and what I regard as the essential values. I shall offer such logic as I can command, supported by such experimental evidence as I know to be available.

The application of limit-design philosophy to trusses lies at the basis of a change of title from that of theory of ductility, which I taught for eight years prior to 1926, to that of limit design. The reason for this change in title may be of some interest.

The last four pages of my book, *Elastic Energy Theory*, (reference 1*h*) are entitled, "Limitations of the Elastic Energy Theory." These four pages may be regarded as the prelude to this book. Even as is done in the book, I have always concluded any course on elastic energy with a discussion of the limitations of the entire theory of elasticity, including the theory of elastic energy. One of the reviewers stated that the book was written with a "crusader's zeal." This may be true, as I did and do now hold that, if we apply the theory of elasticity to the study of strength, the theory of elastic energy is at once the most universal and the simplest of all variations on the theme of elasticity. On one occasion, when I ended a series of lectures on elastic-energy theory with a discussion of the limitations and shortcomings of this theory, one of my colleagues who had attended the course accused me of hitting below the belt and letting my friends down. He, as he said, was rooting for what he called "your theory," when I criticized it more severely than he had ever heard anyone else knock it. This again seems to be a question of sense of value, value this time with reference to professional ethics. I held and hold that in presenting any subject, I am duty-bound to present the reverse side of the medal, if there is a reverse side.

On another occasion, when in 1926 I presented elastic-energy theory as the first engineering course offered by the University of Michigan extension department in Detroit, I again offered at the end of the course my criticism of the theory. On this occasion I had in my audience the late C. M. Goodrich, chief engineer of the Canadian Bridge Company. Mr. Goodrich invited me to Walker-

ville, Ontario, to observe how the Canadian Bridge Company designed transmission towers. When there is what the lawyers call "a meeting of the minds," lengthy arguments are unnecessary. Goodrich had risen to the theory of ductility which I was expounding, like a trout rises to a fly. When he laid before me one blueprint (see Fig. 39 which represents part of such a blueprint and note that in Canada it is customary to represent tension by the minus sign and compression by the plus sign) depicting a tower and its complete strength analysis including dead load, wind load, and torsion, resulting from the breaking of one of the cables, I was at first stunned. I soon realized, however, that the basic principle in the tower design was the same as the one which I advocated in theory of ductility. However, the title of theory of ductility immediately became open to question, as in the buckling of slender columns ductility is not necessarily involved. The title limit design ultimately came to replace theory of ductility.

In a paper on limit design (reference 9) Mr. Goodrich wrote: "The first designs for towers for the Hydro-Electric Power Commission of Ontario were condemned by four eminent authorities, and the weakest of the two first towers, tested in 1910, carried a fifty percent overload." As I understand the early history of the transmission-tower design from Goodrich's own mouth, the method advocated by him was disapproved by the president of the company, by the then chief engineer of the company, by certain consulting engineers, and by the engineering staff of the customer. To settle the argument it was proposed to erect towers in the yard and test them to destruction. This procedure, to my knowledge, is now practiced by all firms on this continent selling transmission towers. It holds special significance, in my view, because it is so rare that we can test and study complete structures to failure. A vast number of tests on columns are of dubious value, because the end conditions actually supplied by a complex structure cannot be simulated in a laboratory. The study of column behavior, therefore, as part of a complex structure like a tower, holds special significance.

In the foregoing quotation from Goodrich's paper, he mentions the date 1910. The sheet from which Fig. 39 is taken carries the date 1907. This very early design was given me by Mr. Goodrich for the purpose of reproduction, because it gives certain details and permits the following of certain steps which, once the designer

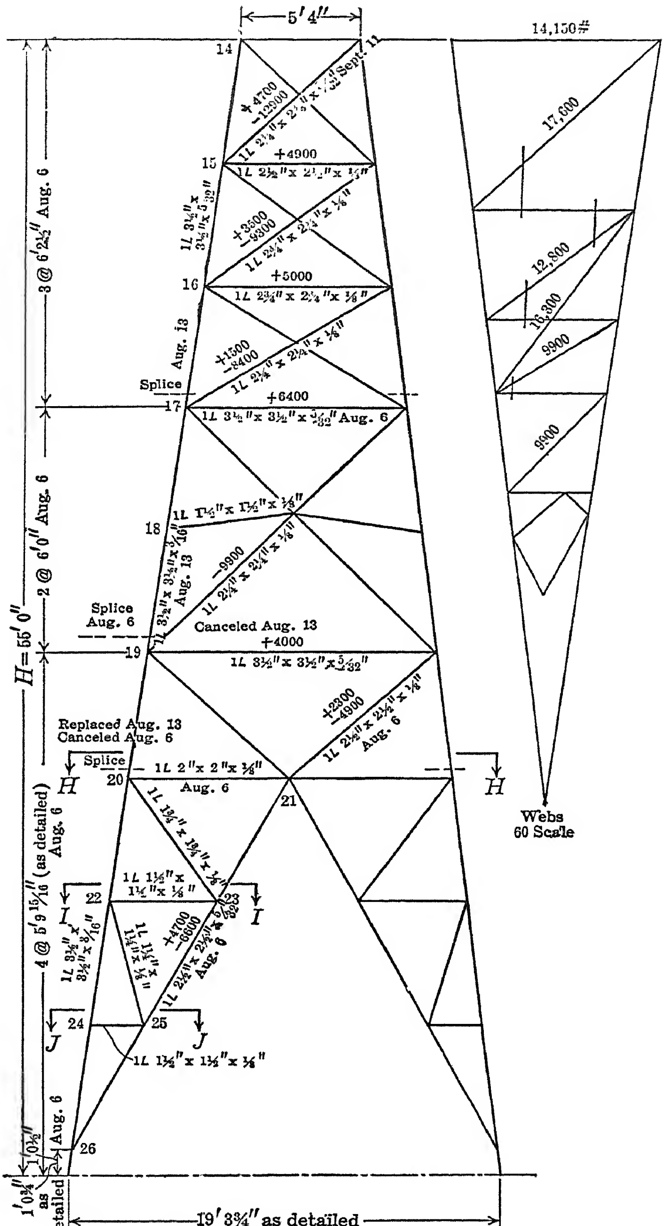


FIG. 39. Transmission-Tower Design.

becomes thoroughly familiar with these steps, are generally not given in later designs.

The theory of limit design is predicated on the characteristic ductile behavior represented by curve *ABC* (Fig. 1). In order that this philosophy may be applicable to the design of trusses, say transmission towers, it is assumed that the behavior of a column, upon buckling, may be closely represented by a similar load-deformation curve. This seems contrary to specifications and to some of the most generally held opinions in regard to column functioning.

ELASTIC COLUMNS

Since we speak of the "confusion" existing with reference to the column problem it may be a worth-while effort to analyze this confusion in the hope of arriving at a little clarification.

First: We have a lack of clarity in the definitions of words. In general, columns are structural compression members. Not all compression members, however, are called columns. The monumental pillars in a temple or in a public building are sure to be called columns. Similarly the vertical compression members in a skyscraper are always called columns. Comparable compression members in a simple railroad truss, however, are called end posts, or top chords, whereas in a derrick they are called booms. The compression members in an airplane are called struts, stringers, or longerons. To add to the confusion, due to inexplicitly defined terminology, some apply the term "elastic stability" to the phenomenon which others would call "elastic instability."

Second: We have as another cause of confusion the apparent inevitable conflict in sense of value. In connection with the slender-column problem we frequently encounter the formulation of a "load–stress relationship" which to many defies interpretation in terms of strength (see page 75). The efforts to extend the stress analysis of columns into the ductile range (reduced-modulus- or double-modulus-column theory) impress me as an overdose of a bad medicine.

It is one thing to point to a possible source of confusion which many seem to believe exists. It is another matter to try to eliminate such confusion. In any efforts to undertake this task, my attempts naturally will be subjective and as prejudiced as any column analysis by a conventional stress analyst. The prejudice,

however, will be of a different brand, and the reader, inevitably, will be confronted with a choice between the two.

As to item one, the possible confusion due to a lack of clarity in terminology, let us begin by formulating our problem. The basic logic involved in the working-stress concept of strength is that, if we design for a working stress equal to one-half the elastic-limit stress, then the load may be increased 100 per cent before the elastic-limit stress is reached—and we thus have a strength factor of safety of 2. This logic implies a linear relationship between load and stress as well as between load and deformation. This is

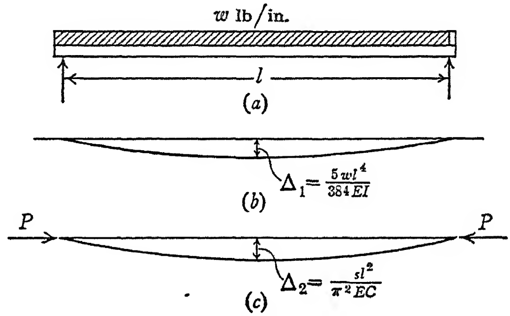


FIG. 40.

an example of what is generally expressed as the principle of superposition being applicable. Thus, in some of our strength problems, if we double the load we double the stresses and the deflections; if we reverse the load we reverse all the stresses and deflections. The distinguishing feature of the slender column is that the load–stress or load–deflection relationships are not linear, and, thus, that the principle of superposition does not apply.

Figure 40*a* represents a simple beam subject to a uniformly distributed load. Figure 40*b* represents the elastic curve of this beam with the central deflection expressed as $\Delta_1 = 5wl^4/384EI$. We ignore this deflection in order to arrive at this value for the deflection. In resorting to the theory of elasticity we express elementary length of the beam as $ds = dx$. We then proceed to integrate along the horizontal X axis instead of integrating along the true elastic curve. We feel justified in doing so, because the error resulting from this procedure is smaller than the numerous errors involved in the determination of the factors w, E, and I, and even of the length l.

If the same beam of Fig. 40*a* is loaded as a column by an axial load P only, then the maximum stress in the column is given as $s = P/A + Mc/I = P/A + P\Delta_2 c/I$. The elastic curve (Fig. 40*b*) will be expressed as a fourth degree equation, while the elastic curve (Fig. 40*c*) will be a sine curve. If the Δ's in both instances correspond to the same maximum stresses, then these Δ's will be of substantially identical magnitudes. It seems clear that, although we were justified in ignoring the Δ in the analysis of the beam, it does not follow that we will be equally justified in ignoring it in the column. In Fig. 40*c*, Δ_2 is proportional to the length square. Thus for long and slender columns we are not justified in ignoring the deflections when we set out to determine the strength of the column, whereas for short and stubby columns we are justified in ignoring it. If we ignore it we substantially say we assume the principle of superposition to apply, and when we assume the principle of superposition to be applicable we may analyze the column in like manner as we analyze beams.

Eccentricity. It is generally recognized that centrally applied loading of columns is unrealizable in practice and is very difficult to attain even under carefully controlled laboratory conditions. It is thus held by many that any column formula, to be at all

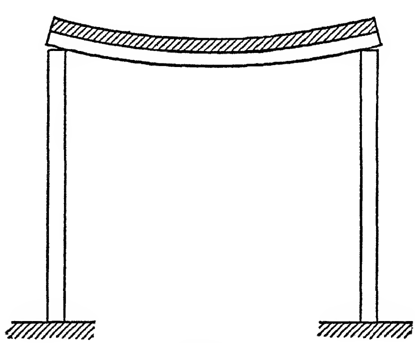


FIG. 41.

rational, should contain the symbol e to give expression, in some manner, to this eccentricity effect. I hold that this has, as yet, never been accomplished, and, further, I see small likelihood that it will ever be accomplished. My reasons for this view are as follows:

If a simple beam is supported on two slender columns (Fig. 41), and loaded with a uniformly distributed load, then, as the beam

assumes a curved shape, it will ride on the edge of the bearing plate, and the columns will be loaded by what we may call a positive eccentricity. Once the columns buckle, if the columns are slender, then the curvature of the columns at their tops will be greater than that of the beam; the beam will then ride on the outer edge of the bearing plate, and the eccentricity will have changed from a weakening positive one to a strengthening negative one. Metal structures are generally bolted, riveted, or welded. If we assume the beam of Fig. 41 to be welded to the columns, then, as the beam is loaded, a moment will be introduced at the top of the column. The bending of the column will thus take place gradually instead of suddenly. As the bending of the slender column progresses, the moment or the eccentricity will gradually decrease, will finally change sign, become negative, and become a strengthening instead of a weakening factor. The point which I am trying to make is that eccentricity is inherently a variable and not a constant. It can be ascertained with any degree of precision only at the end of a complicated analysis, and not at the beginning of such analysis.

Figure 42*a* represents a doubly symmetrical structure, consisting of two beams supported by two columns. If the beams and columns are assumed to be pin-connected, then the elastic curves appear as shown by the dashed lines in Fig. 42*b* where the angle ϕ_2 represents the angular deflection of the column corresponding to the elastic-limit stress being reached at the middle point of the column. If the columns are slender, then ϕ_2 will be larger than ϕ_1. If the columns are slender and rigidly connected to the beams, then the elastic curve of the column will appear as shown in Fig. 42*d*. The moments at the ends of the columns, or, what amounts to the same thing, the eccentricities, will have changed sign and will have become strengthening instead of weakening effects. A detailed mathematical analysis of the problem represented by Fig. 42 may be found in reference 1*e*. The conclusions of this analysis are graphically represented in Fig. 43. In this figure $n = L/l$, where L is the length of the column between points of inflection and l is the geometric length of the column (Fig. 42*d*). The open circles represent values computed on the assumption that the principle of superposition is applicable (the vertical legs of the culvert are treated as beams, and the analysis is made by a method like area moments, end-moment distribution, or elastic-energy theory). The solid lines in Fig. 43 represent values computed on the assumption that the principle of superposition is not applicable.

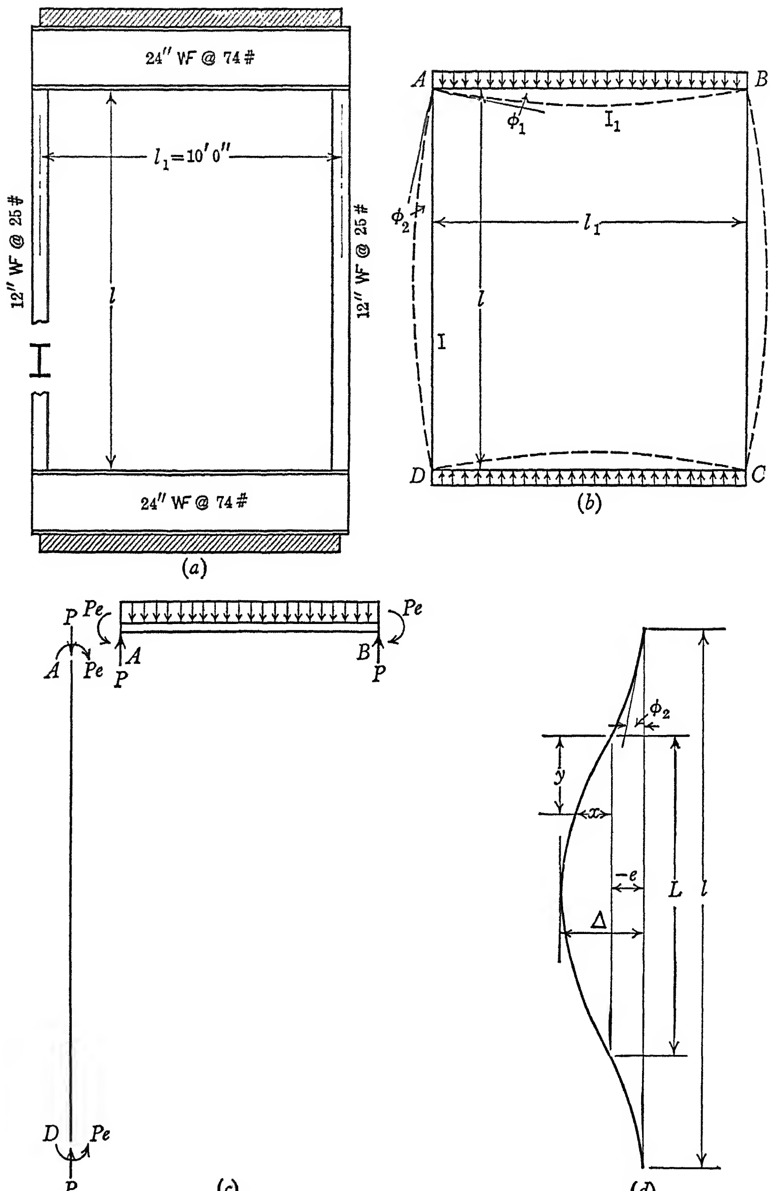

24″ WF @ 74 #
$l_1 = 10'0''$
12″ WF @ 25 #
l
12″ WF @ 25 #
24″ WF @ 74 #
(a)
A
B
ϕ_1
I_1
ϕ_2
l_1
l
I
D
C
(b)
P
Pe
A
Pe
P
A
Pe
B
P
D
Pe
P
(c)
ϕ_2
y
x
e
L
l
Δ
(d)

Fig. 42.

The dashed curve in Fig. 43*a* represents the Euler values for column *AD* (Fig. 42) based on the assumption that the column is ideally pin-connected.

It appears from Fig. 43*c* that, if the principle of superposition is assumed to be applicable, then the eccentricity decreases as

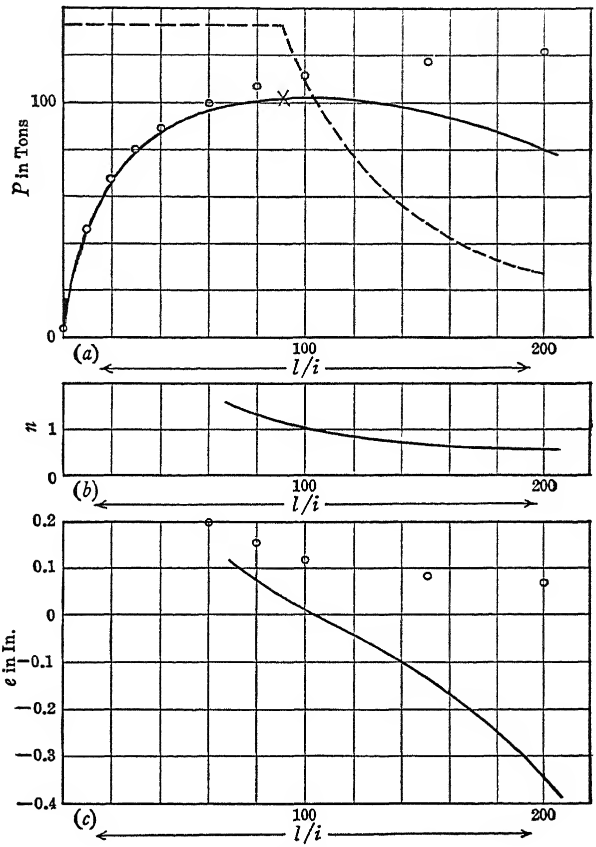


FIG. 43.

l/i increases and approaches the value $e = 0$ for the case l/i approaching infinity. According to the more correct analysis, based on the assumption that the principle of superposition is not applicable, e is positive for values of $l/i < 103$ and becomes negative for values of $l/i > 103$. As will be argued presently, n and e represent substantially the same thing, and, as may be judged from Fig. 43*b*, the value of n is greater than unity for values of

$l/i < 103$ and becomes less than unity for values of $l/i > 103$. Similarly, e is positive for values of $l/i < 103$ and becomes negative for values of $l/i > 103$. The most significant curve, in our present argument, is Fig. 43*a*. It appears that for short columns the approximate method, if it is assumed that the principle of superposition applies and further that the proportional-limit stress s_1 determines the strength of the column, gives substantially the same result as the more exact method which assumes that the principle of superposition is not applicable. The discrepancy between the exact and the approximate method of analysis, for columns in the short-column range, if the latter is used, is not on the safe side, but is small enough to be absorbed in the factor of safety. For values of l/i in the long-column range, the discrepancy between the exact and the approximate methods of analysis is of considerable magnitude.

It is of interest to note from Fig. 43*a* that, based on either the exact or the approximate method just discussed, both of which are elasticity methods, the strength of the column increases as l/i increases until the maximum strength of the column is reached for a value of $l/i = 100$. Limit-design reasoning would attribute a greater strength to the column for values $l/i < 100$ than those indicated on Fig. 43*a*.

The reader is reminded that the discussion of the last pages is an attempt to clarify a possible source of confusion due to inexplicit terminology. Many hold that the true column is a compression member so slender that, if analyzed by the theory of elasticity, it is necessary to assume that the principle of superposition is not applicable. In this class fall many structural members which are never called columns. All members in the transmission tower must be designed as compression members. Whether they are loaded in tension or compression depends on the direction in which the wind may blow and the side of the line on which a cable may break. Without the cross bracing the legs would not carry their own weight. The purpose of the cross bracing is to reduce the effective length of the legs as columns. The cross member, the diagonal bracing, and the legs, thus, are all columns. The bulkheads in an airplane wing serve the same purpose as the diagonal bracing in the tower, and what we commonly call stringers are thus also columns. The end posts in a bridge, or the columns in a building, ordinarily are not columns in the sense as defined here.

Our primary interest lies in columns in the long-column range. Compression members in the short-column range may be analyzed, if we use elasticity methods, by such formula as $s = P/A \pm Mc/I$, by end-moment distribution, and the like, or by using limit-design methods, as illustrated in the earlier part of this book.

One elasticity column formula involving the eccentricity term e is the following:

Eccentricity or Wow (Initial Crookedness). Assume a column to be initially curved in the shape of a complete arch of a sine curve with an initial offset and radius of curvature at the midpoint equal to e and R, respectively (Fig. 44). As the column is loaded with a load P the deflection will increase an amount Δ. The bending-moment area is one half of the area bounded by the line of action of the load and the elastic curve. The maximum ordinate of this bending-moment area is at the midpoint and equals $P(e + \Delta)$. The deflection $\Delta = \text{area}\,\bar{y}/EI$. The area in question, shown cross-hatched in Fig. 44, is $2/\pi \times$ the circumscribed rectangle. Thus

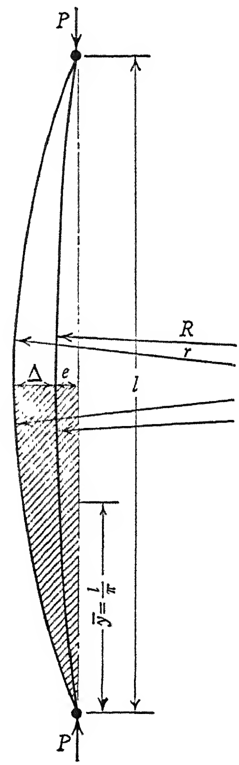


FIG. 44.

$$\text{Area} = \frac{2}{\pi} P(e + \Delta) \frac{l}{2}$$

The $\bar{y}$ of an area under a sine curve, shown in Fig. 44, is l/π. Therefore

$$\Delta = \frac{\text{Area}\,\bar{y}}{EI} = \frac{Pl^2}{\pi^2 EI}(e + \Delta) = \frac{Pl^2\Delta}{\pi^2 EI} + \frac{Pl^2 e}{\pi^2 EI}$$

or

$$\left(EI - \frac{Pl^2}{\pi^2}\right)\Delta = \frac{Pl^2 e}{\pi^2} \qquad \text{Equation m}$$

From the theory of strength we have

$$\frac{1}{r} - \frac{1}{R} = \frac{M}{EI} = \frac{s}{cE}$$

$$\frac{d^2x}{dy^2} = \frac{1}{r} = \frac{\pi^2(\Delta + e)}{l^2}$$

$$\frac{d^2x}{dy^2} = \frac{1}{R} = \frac{\pi^2 e}{l^2}$$

$$\frac{1}{r} - \frac{1}{R} = \frac{\pi^2\Delta}{l^2} = \frac{s}{cE}$$

The stress due to the increase in curvature therefore is

$$s = \frac{\pi^2 \Delta c E}{l^2}$$

This stress is augmented by the stress P/A due to the direct load.

Thus the resulting extreme fiber stress on the inside of the curve at the midsection of the column is

$$s = \frac{\pi^2 \Delta c E}{l^2} + \frac{P}{A}$$

or

$$\Delta = \left(s - \frac{P}{A}\right)\frac{l^2}{\pi^2 c E} \qquad \text{Equation n}$$

Eliminating Δ between equations (m) and (n), we obtain

$$\left(EI - \frac{Pl^2}{\pi^2}\right)\left(s - \frac{P}{A}\right) = PecE$$

or

$$P^2 - \left(\frac{\pi^2 EI}{l^2} + sA + \frac{\pi^2 EecAi^2}{l^2 i^2}\right)P + \frac{\pi^2 EIsA}{l^2} = 0$$

If we let $\pi^2 EI/l^2 = P_{cr}$, then the solution of this equation gives

$$P = \frac{1}{2}\left\{\left[sA + P_{cr}\left(1 + \frac{ec}{i^2}\right)\right] \pm \sqrt{\left[sA + P_{cr}\left(1 + \frac{ec}{i^2}\right)\right]^2 - 4P_{cr}sA}\right\}$$

or

$$\frac{P}{A} = \frac{1}{2}\left\{\left[s + \frac{\pi^2 E}{(l/i)^2}\left(1 + \frac{ec}{i^2}\right)\right] \pm \sqrt{\left[s + \frac{\pi^2 E}{(l/i)^2}\left(1 + \frac{ec}{i^2}\right)\right]^2 - 4\frac{\pi^2 E}{(l/i)^2}s}\right\}$$

The last two expressions give a relationship between the load P and the maximum stress in the column. This relationship is purely academic and of no interest to the designer. If we define s as being equal to the elastic-limit stress s_1, P immediately assumes special significance, because it then becomes the symbol for the maximum load which the columns can carry, if we accept the traditional assumption that the elastic-limit stress s_1 is our criterion of strength. The resulting formula for the load P then becomes

$$\frac{P}{A} = \frac{1}{2}\left\{\left[s_1 + \frac{\pi^2 E}{(l/i)^2}\left(1 + \frac{ec}{i^2}\right)\right] \pm \sqrt{\left[s_1 + \frac{\pi^2 E}{(l/i)^2}\left(1 + \frac{ec}{i^2}\right)\right]^2 - \frac{4\pi^2 E s_1}{(l/i)^2}}\right\} \qquad \text{Formula III}$$

In this formula s_1 represents the elastic-limit stress, and P represents the load which will induce this elastic-limit stress. Formula III is avowedly slightly approximate for all values of e, except for the value of $e = 0$, when the formula is exact.

Another elasticity-column formula involving the eccentricity term e is the secant formula:

$$s = \frac{P}{A}\left(1 + \frac{ec}{i^2}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right) \qquad \text{Formula IV}$$

In rational-column analysis (reference 1*e*) I called this formula IV exact. For purposes of design I also called it futile. I call it exact, because I believe that the philosophy on which it is based is thoroughly sound and rational. I thus believe that it represents a relationship between load and stress as exact as does any elasticity formula. When we come to solve the formula, we find it is not so exact as it appears, because we must begin by substituting a value for s, and then, by trial and error or by successive approximations, seek a value for P which will satisfy the formula. Since the process of successive approximations is inexact, it leads to the conclusion that although the formula itself may represent an exact relationship, the solution of the formula is inherently inexact.

I called the formula futile, because it involves the assumption that the principle of superposition is inapplicable. Therefore, to

apply it to columns in the short-column range, where we may have a chance to determine the value of e, seems inconsistent or at least inadvisable. These remarks apply with equal force to

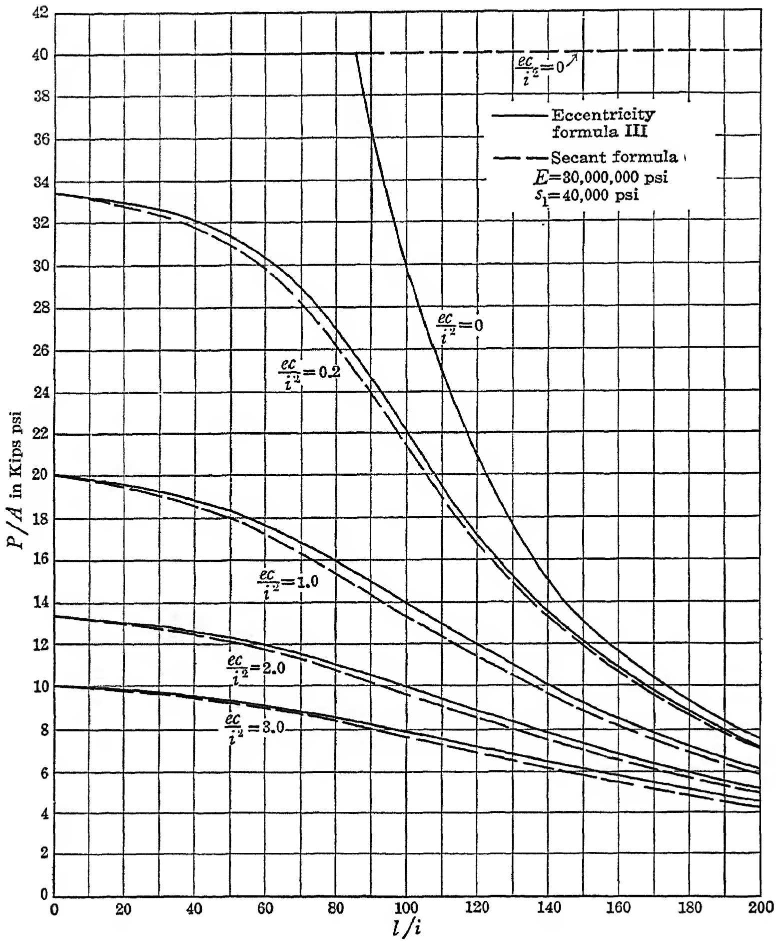


FIG. 45. Graphical Comparison between Wow Formula and Secant Formula.

formula III. Both formulas III and IV give substantially the same results (see Fig. 45). Their accuracy is attested to by Fig. 46, which we offer as a check by experimental evidence of a theoretical formula. A satisfactory check of a formula in a laboratory does not give the formula any special value as a design formula. In a

laboratory we are able to control eccentricities. Until we have some easy way for determining the values for e which are to be substituted in the formula, the value of the formula remains

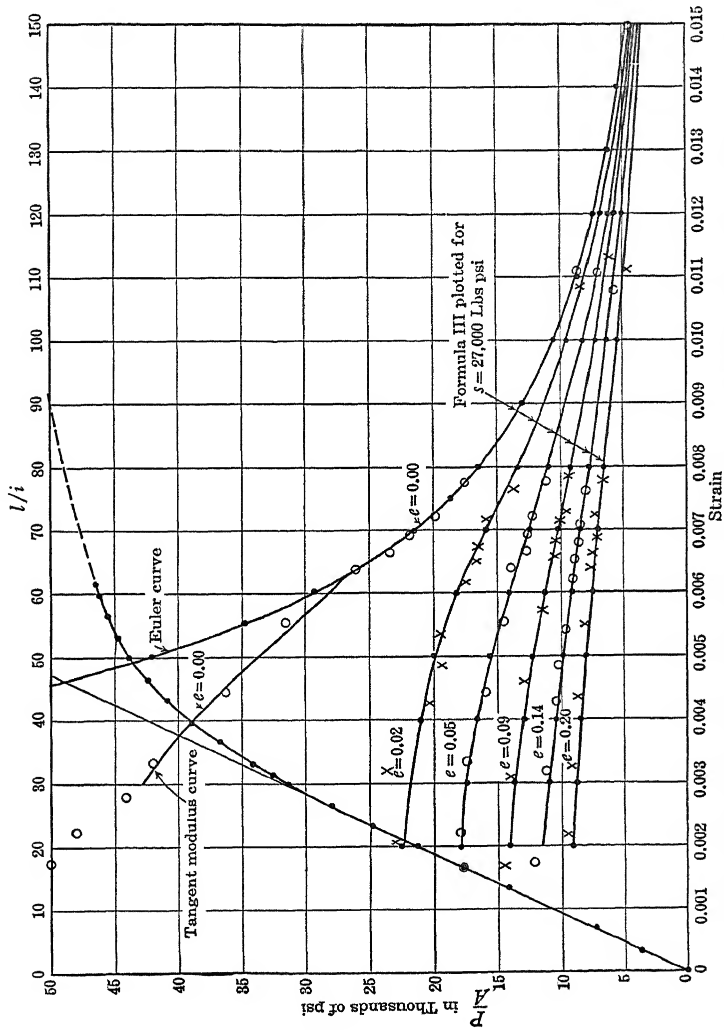


FIG. 46. Experimental Verification of Wow Formula and Tangent Modulus Formula. Columns were $\frac{5}{8}''$ square 24 ST aluminum extrusion.

purely academic. As such, formula III has significance. If the value $e = 0$ is substituted in formula III it simplifies to:

$$\left.\frac{P}{A}\right] = \frac{1}{2}\left\{s_1 + \frac{\pi^2 E}{(l/i)^2} \pm \sqrt{\left[s_1 - \frac{\pi^2 E}{(l/i)^2}\right]^2}\right\} \quad \text{Formula V}$$

This is a special form of Euler's equation, which, however, gives two values for P/A. Thus Euler's equation appears, not as a continuous curve with P/A approaching infinity as l/i approaches zero, but as a combination of two curves (see Fig. 47). For the short-column range, formula V gives $P/A = s_1$, and for the long-column range it gives $P/A = \pi^2 E/(l/i)^2$. Figure 48 offers experimental support for formula V. (For formulas III, IV, and V, and Fig. 47 see reference 1*e*; for Fig. 46 see reference 1*j*.)

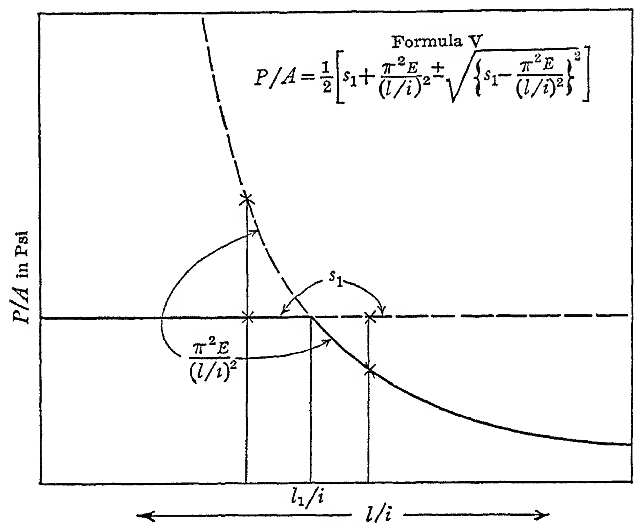


FIG. 47. Graph of Formula V.

The second possible source of confusion, of talking or writing at cross purposes, is apparently an unavoidable conflict in objectives and sense of value. In an attempt at clarification I can do little more than define my own objective and state my own sense of value. My interests in columns are:

1. I am interested in the functioning of columns as integral parts of complex structures (cross bracing in transmission towers and stringers in airplanes).

2. I am interested in predicting the maximum load-carrying capacity of such columns.

3. Another of my interests lies in the determination of the sustaining power of such columns, once the maximum load has been reached or exceeded. In other words, I am interested in the load-

deformation relationship in the direction of the load, after buckling has commenced.

My engineering interest in load–stress relationships or in test results is limited to the extent to which such relationship may throw light on the three propositions listed.

In theoretical analysis two types of columns are frequently considered, the fixed-ended and the pin-ended column. Perfect

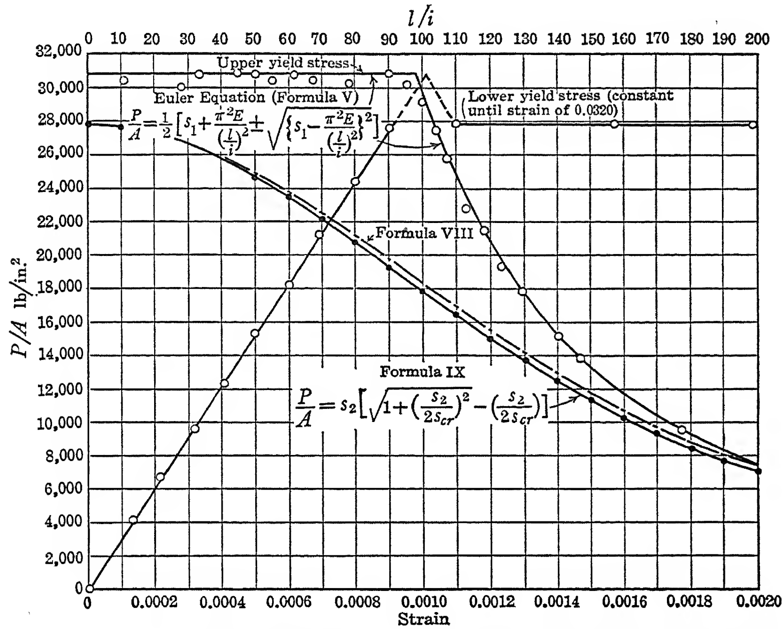


FIG. 48. Experimental Verification of Formula V and Graphs of Formulas VIII and IX.

fixity in practice, it is believed, is nonexistent; neither, to my knowledge, is perfect fixity realizable in the laboratory. Pin-ended conditions of support are realizable in the laboratory. In practice we have pin-ended conditions when we consider that the points of inflection, which are points of zero moments, may be likened to the ideal pin end. In Fig. 42*d*, *L* marks the length of the column between its points of inflection, the points of zero moment, or the equivalent of pin ends. As seen from Fig. 43, $n = L/l =$ unity for the case when $l/i = 103$. For this slenderness ratio the theoretical pin-ended length of the column equals that of its geometric length. When $l/i < 103$ the theoretical

points of inflection, or pin ends, fall beyond the actual length of the column.

In 1729 P. van Musschenbroek discovered experimentally the fact that the strength of wooden columns varies inversely as the square of their length. In 1757 (reference 10) Euler developed theoretically a quantitative expression for this law:

$$P = \frac{\pi^2 Ekk}{l^2} \qquad \text{Euler's equation}$$

This early contribution by Euler is one of the classics in engineering literature. Note that no symbol for stress appears in the Euler equation. It is strictly a limit-design formula. In fact, in Euler's day, the concept stress was not formulated, and stress analysis was not even dreamt of, or at any rate, it had not made its appearance anywhere in print. The term Ekk is what we now write as EI. Euler named it "moment du ressort" or "moment de roideur." By E he represented what he called the quality factor (Young's modulus was not formulated until 1807), and by the term kk he represents a dimensional factor. Of the term kk he says, for cylindrical columns: "Son moment de roideur seroit proportionnel au cube, où peut être plutôt au quarré quarré du diametre." He suggests that the value for Ekk may be obtained experimentally from the formula $\Delta = Pl^3/3Ekk$, the deflection of the end of a cantilever subject to a concentrated transverse load at its end. He develops this expression apparently for the purpose of using it to find the term Ekk, on the mere assumption that the intensity of the curvature at any point in a beam is proportional to M/Ekk. The development of this deflection formula seems almost as ingenious as the development of the Euler equation.

By taking Euler's original equation,

$$P = \frac{\pi^2 EI}{l^2}$$

and dividing it by the factor A we obtain

$$\frac{P}{A} = \frac{\pi^2 E}{(l/i)^2}$$

Since in formula I, P/A was called stress, some stress analysts insist on calling P/A stress also in this expression. It matters not a great deal what we call it so long as we realize that P represents

a failure load which means stresses equal to or greater than the proportional-limit stresses. Thus, P/A, as stress, is nowhere to be found in the column.

Euler's equation, on the one hand, has long been championed as the essence of theoretical column analysis; on the other hand, it has been under suspicion on two counts. First: For many years it defied satisfactory experimental verification. For the last 40 years this criticism may be considered as being dispelled, witness Fig. 48. Second: Although the Euler equation may be a sound theoretical expression based on certain assumptions, these assumptions do not exactly fit actual service conditions. This last suspicion finds expression in the fact that, although we may have a variety of column-design formulas, all of which give values that vary widely from each other and few of which lay any claim to rationality, they nevertheless all agree in giving values for column strength in the intermediate-column range,* which are considerably smaller than those obtained by means of Euler's equation.

The foregoing pages are an attempt to present the column problem, possibly somewhat involved, but, it is hoped, free from confusion and contradictions. Before closing this chapter I should like to present a consideration which is intended to be a suggestion towards simplification.

I have stressed throughout this book, where occasion seemed to demand it, the distinction between the primary and the secondary. I repeat that in engineering mechanics the laws of equilibrium are primary; all else is secondary. One of the most important conclusions derived from the laws of equilibrium is the dictum that the resultant of a couple and a force is a single force. The line of action of a load eccentrically applied to a column passes through the point of zero moment, the point of inflection, the perfect pin-end condition. This point of zero moment may fall either within the boundaries of the column or beyond them. In terms of Figs. 42 and 43, the end moments of the columns would be of a sign as indicated by the arrows in Fig. 42*c* for relatively small values of l/i. As l/i based on the actual length of the columns increases, the magnitudes of the end moments or the eccentricities decrease until they become zero and change sign. Correspondingly, the theoretical length L decreases. At the beginning of loading L equals infinity. When the end moments

* For a definition of intermediate-column range see page 83.

and the eccentricity e approach zero, L approaches the value of l. When the end moments and the eccentricities change sign, then the length of the pin-ended portion of the column L becomes less than the geometric length l (Fig. 42*d*).

Consider also Fig. 49, which represents to an exaggerated scale the outline of a built-in column. The built-in column is as unrealizable in practice or in the laboratory as is the beam built-in at both ends. It will serve our purpose for an academic discussion, because, presumably, every engineer is familiar with it from the time he first studied the column problem. As to what makes the column (Fig. 49) bend to the left instead of to the right—this is probably due to a slight initial wow, or to a slight eccentricity. Once the column buckles, the eccentricity passes through zero and becomes negative. The force and moment shown at the bottom of the column (Fig. 49) are the exact equilibrant to the force which is offset with an eccentricity e shown at the top. As the column continues to bend, the eccentricity e shown at the top of Fig. 49 increases, or, what amounts to the same thing, the moment shown at the bottom increases. The value of P remains constant in this special case. The ratio, $n = L/l = 0.5$, also remains constant in this special case.

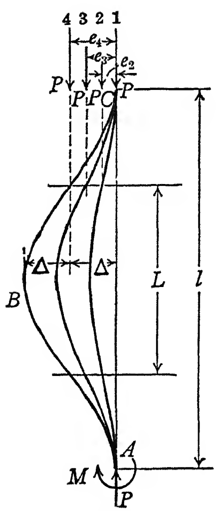


FIG. 49.

The main theme of "Rational Column Analysis" (reference 1*e*) was to call attention to the identity of eccentricity, end moments, and the ratio $n = L/l$. The identity of force and end moment with eccentricity is the direct consequence of the laws of equilibrium. These laws are the most absolute laws we have in engineering mechanics, regardless of how they may be appraised in the field of celestial mechanics. The intimate relation between e and n (Fig. 43) follows from the laws of elasticity. Within the sphere of elastic behavior (reference 1*e* is limited to consideration of elastic behavior), eccentricity and end moments are identities, which are intimately related to n. That is, if we know one we know the others.

In Fig. 49 the coefficient n may be accurately appraised, whereas the end moments or eccentricities are variables, not only in one

particular column, but also from column to column. In Fig. 42 the answers are not so explicit. Ultimately we may have to resort to an exercise of our judgment, to an intelligent estimate. When that becomes necessary I am inclined to believe that we can make a more satisfactory estimate of n than we can of either e or the end moments. An emphasis on end fixity, or on coefficient $n = L/l$, and a corresponding avoidance of eccentricity or end-moment consideration would seem to be in the direction of clarification, or simplification, of the column problem.

LIMIT DESIGN OF COLUMNS

In the present discussion it becomes necessary to introduce a few new terms:

1. Intermediate-Column Range. Most engineers commonly differentiate between columns "in the short-column range" and those "in the long-column range." We like to make a further differentiation and speak of columns in "the intermediate-column range." In terms of the steel-column curve, Fig. 48, we define the intermediate-column range as extending from $l/i = 70$ to $l/i = 120$. In terms of the aluminum-column curve, Fig. 50, the range extends from $l/i = 45$ to $l/i = 80$.

2. Critical Column Length and Critical Slenderness Ratio l_1/i. By *critical column length* we mean l_1 as defined by the equation: $\dfrac{P_1}{A} = \dfrac{\pi^2 E}{(l_1/i)^2}$, in which P_1/A equals proportional limit stress. By *critical slenderness ratio* we mean l_1/i, as defined by the same equation. In Fig. 48 the critical slenderness ratio is $l_1/i = 97$, whereas in Fig. 50 it is 64.

3. Elastic Stability. Figure 51 represents a laboratory test record of the average load–axial deformation relationship of $1\frac{1}{2}'' \times 1\frac{1}{2}'' \times \frac{1}{8}''$ steel angles, centrally loaded as columns (for centering of loads see references 1*e* and 1*i*, or 1*m*). Figure 52 represents stress–strain curves for such materials as steel and aluminum. Figure 53 represents stress distribution over the critical cross-section area of a column during various stages of buckling, and Fig. 54 shows an elastic slender column after it has buckled.

We would like to direct special attention to the curve for $l_1/i = 306$ in Fig. 51. As the column is loaded to an average load P_2/A, represented by B in Fig. 51, or by A in Fig. 52, then,

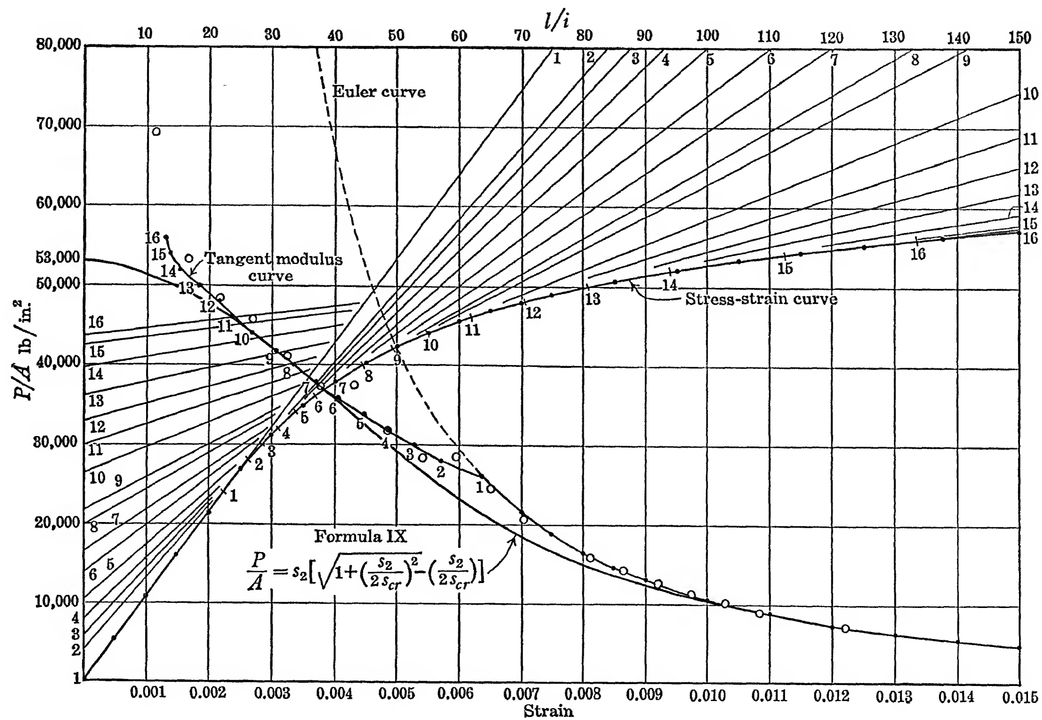


FIG. 50. Experimental Verification of Tangent Modulus Curve and Graph of Formula IX. Columns were $\frac{3}{4}''$ round 24 ST aluminum extrusion, cold-stretched 1.95 per cent and aged 20 weeks.

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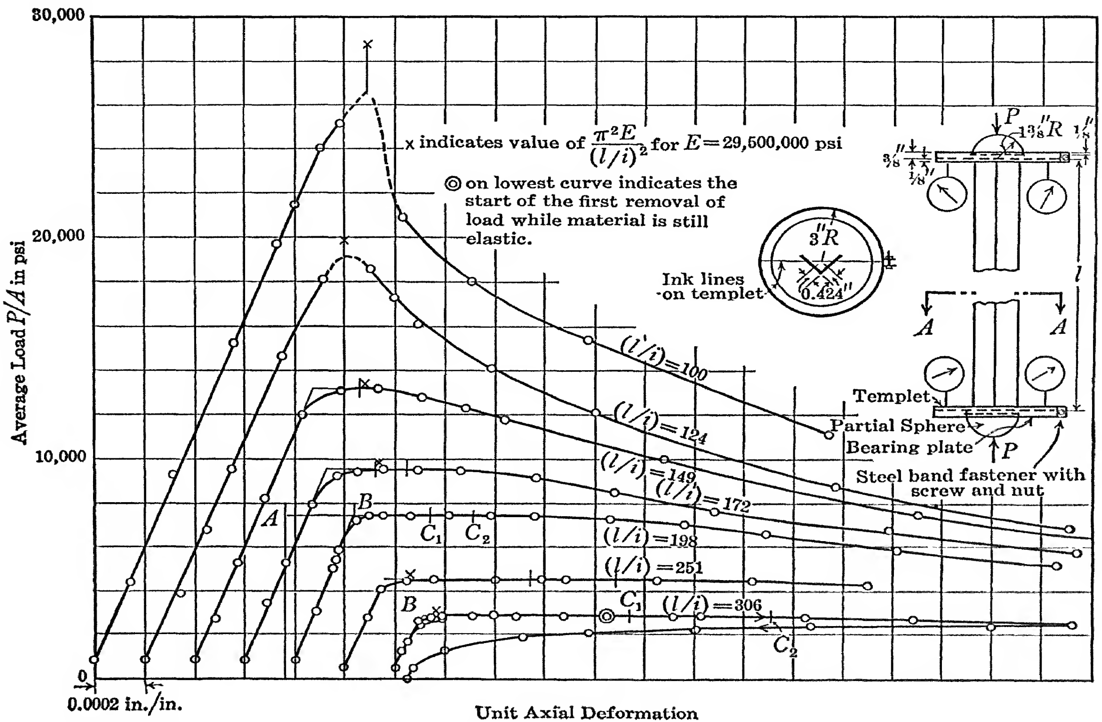


FIG. 51. Load–Axial Deformation Curves for $1\frac{1}{2}''$ x $1\frac{1}{2}''$ x $\frac{1}{8}''$ Mild Steel Angles Concentrically Loaded.

if the load is centrally applied and the column is straight, the shortening of the column due to direct compression is represented by a in Fig. 54, and the stress distribution over the cross-section

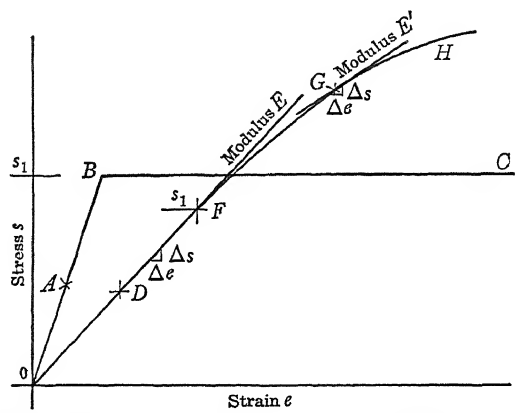


FIG. 52. Representative Stress–Strain Curves for Mild Steel and for Aluminum Alloy.

area is represented by Fig. 53a. The column now buckles and assumes a sidewise deflection Δ and a corresponding curvature. The distance b, Fig. 54, represents the linear deformation of the column due to the elastic bending of the column in the shape of

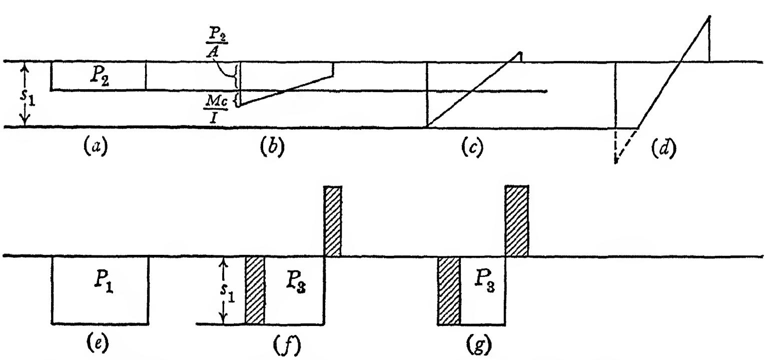


FIG. 53. Changes in Stress Distribution as a Mild-Steel Column Buckles.

a sine curve. This distance is represented in Fig. 51 as BC_1 or BC_2. Within the elastic range the stresses are a function of the curvature, and thus the stress distribution after buckling appears as Fig. 53b.

Euler's basic assumption was that the deflection Δ is proportional to the curvature, and this, in turn, is proportional to the resisting moment. We now know that the bending stresses also are proportional to this resisting moment. Thus the bending stresses are proportional to Δ.

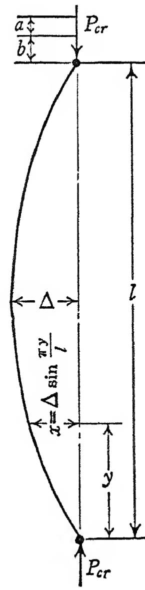


FIG. 54.

$$s = \frac{P}{A} \pm \frac{Mc}{I} = \frac{P}{A} \pm \frac{P\Delta c}{I} \qquad \text{Equation p}$$

The axial deformation b resulting from the elastic bending of a buckled column is

$$b = \frac{\pi^2 \Delta^2}{4l} \qquad \text{Equation q}$$

which, by simple transformation (see page 10, reference 1e), becomes

$$b = \frac{(s_1 - s_{cr})^2 l^3}{(2\pi c E)^2}$$

or

$$\frac{b}{l} = \left[\frac{(s_1 - s_{cr})l}{2\pi c E}\right]^2 \qquad \text{Formula VI}$$

in which $s_{cr} = \dfrac{\pi^2 E}{(l/i)^2}$. An elastic bar subject to an average tensile force smaller than s_1 is in equilibrium and presents a case of *elastic stability*. Similarly, an elastic bar, centrally loaded and subject to an average compressive force smaller than $\dfrac{P}{A} = \dfrac{\pi^2 E}{(l/i)^2}$, is in equilibrium and presents a case of *elastic stability*. (All values of P/A less than that represented by point B in Fig. 51.)

4. Elastic Indifferent Equilibrium. The point on the curve for $l/i = 306$ in Fig. 51, shown by the two concentric circles, represents a case of elastic indifferent equilibrium. That is, we may vary the deformation without affecting the load P. However, if we vary the load P we immediately affect the deformation very materially. The testing machine is a machine by which we control deformations, not loads. We obtain loads only when we place a test specimen between the cross heads of the machine and proceed to vary the distance between the cross heads. When the test

record for curve $l/i = 306$ (Fig. 51) was obtained the machine was reversed; that is, the deformations were decreased after the reading represented by the two concentric circles was taken. The record of the load–deformation relationship for decreasing deformations proved to be identical with that for increasing deformations, except for a few experimental points in the region of point B. That is, under decreasing deformations the values for P/A first remained constant, falling on the horizontal line C_2B, and then on the inclined line from point B to the origin.

The values for P/A and unit deformation b/l, represented by the horizontal portion of the $P/A:b/l$ curve (Fig. 51), distance BC_2, on curve $l/i = 306$ or on curve $l/i = 198$, illustrate *elastic indifferent equilibrium.* This question of elastic indifferent equilibrium involves what in mathematical parlance we mean by dependent and independent variables.

Suppose we have a water tank nicely centered on a slender column and proceed to pour water slowly into the tank. When a certain load, $P = \pi^2 EI/L^2$, is reached, the column begins to go down. Theoretically, and I dare say actually, if we have matters very nicely under control, a condition of elastic indifferent equilibrium may be obtained when under this load P the column can assume a wide range of deformations. If however, the load P is increased by merely a few pounds, the distance BC_2 (Fig. 51) is obtained suddenly. The column then continues to deform under a decreasing load, which means that under the water tank filled with a constant load equal to P it would suddenly and completely collapse.

The question of elastic indifferent equilibrium is analogous to the question of ductile indifferent equilibrium, which we discussed on page 3. In a tension bar of mild steel, if the load is the independent variable, the deformation BC (Fig. 1) takes place suddenly. If the deformation is the independent variable, as it is in a testing machine, then several simultaneous values for deformation and load (a constant load) may be obtained.

Suppose we consider next a truss as shown in Fig. 55a. The truss is geometrically symmetrical with reference to the vertical center line. If we assume the vertical legs as well as the diagonal bracing to be identical in size, then the truss is also elastically symmetrical. It may be considered as composed of two trusses (Figs. 55b and c) functioning to the same end, that of supporting the load P_1. Thus Fig. 55a, represents a redundant truss. Elastic considerations tell us that, if bar a were infinitely rigid, the stresses

in c and d would be identical, because the deflections of point A in Fig. 55b would be identical to that of point B in Fig. 55c. If bar a shortens under load, then the stresses in bar c, under elastic behavior, are larger than those in d, because point A in Fig. 55b deflects more than does point B in Fig. 55c. If the strut c buckles, then, according to dictum 3, the deformations of the system remain of the order of magnitude of elastic deformations. The deformation of strut c, therefore, is the independent variable, since it is controlled by the elastic functioning of the remaining components of the truss, and the load resisted by the strut c is the dependent

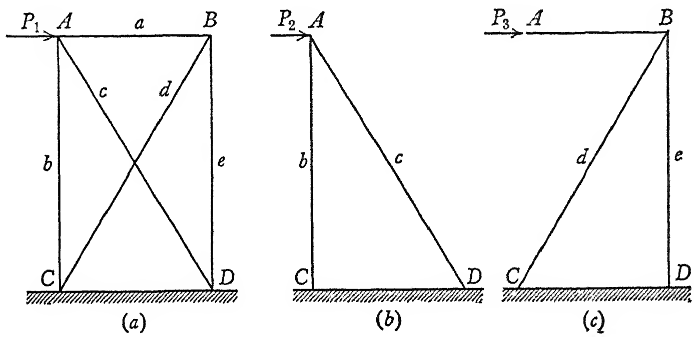


FIG. 55.

variable. Strut c thus behaves like a column in a testing machine, but unlike the column supporting a water tank. If the load P_1 is increased after buckling of strut c has commenced, the resistance of strut c, that is, the resistance of the truss represented by Fig. 55b, remains frozen, while succeeding load increments are absorbed by the truss represented by Fig. 55c.

The maximum elastic deformation resulting from the bending of the column is shown as b in Fig. 54, and as BC_1 or BC_2 in Fig. 51 and is given by formula VI. It may be observed from formula VI that b is a function of c, the distance from the neutral axis to the extreme fiber. The distance b, for equal legged angles such as were used in the tests, thus depends on how the column buckles, whether, after buckling, the outstanding legs are in compression or in tension. In the test this fact was not recorded. This explains the marking of two distances BC_1 and BC_2 in Fig. 51 to represent the equivalent of b in Fig. 54.

Formula VI shows that either b in Fig. 54 or Bc in Fig. 51 depends on $(s_1 - s_{cr})$ and thus is larger in slender columns than in

relatively short columns. It becomes zero in case the columns are of critical length l_1 or shorter. This appears to be well substantiated by Fig. 51. It also seems self-evident from the consideration of Fig. 53. The smaller the value of P_2/A when buckling commences (Fig. 53*a*), the more elastic deformation, involving successive stages of stress distribution represented by Fig. 53*b*, is possible before the critical stage of Fig. 53*c* is reached. In order that the elastic stress moment may continue to balance the external moment $P\Delta$, the stress pattern on Fig. 53*d* should appear as that with the small dashed triangle included. The ductility of the metal, sooner or later, prescribes a stress pattern for Fig. 53*d* without the small dashed triangle. The internal resisting moment and the external moment $P\Delta$ thus fail to balance as they do during successive changes from Fig. 53*a* to Fig. 53*c*. This can only mean that the load P must decrease with increasing deformations once the elastic limit stress s_1 is exceeded.

5. Ductile Equilibrium. If a pin-ended column of critical length l_1 or shorter is centrally loaded, the stress distribution shown on Fig. 53*e* is reached just before buckling commences. Buckling means that an offset Δ (Fig. 54) is suddenly introduced. This means, then, that the stress pattern at the critical cross section of the column must account for a force P_3 and a moment $P_3\Delta$. If stresses are unable to increase beyond the value of s_1 the need for a stress pattern qualitatively equivalent to that shown in Fig. 53*f* is imperative. I concede that sharp corners are rare in nature and that the inner corners of Fig. 53*f* will probably be slightly rounded. I believe, however, that the stress pattern represented by Fig. 53*f*, a pattern very nearly attained, is as reasonable as the stress pattern shown by Fig. 14*g* or 17*c*. As to the stress patterns of Figs. 14*g* and 17*c*, we claim that they may actually resemble that of Fig. 18*a*, but that this will not affect materially any computations based on them. The same reasoning might be applied to Fig. 53*f* or 53*g*. The buckled column is still in equilibrium if we regard the deformation as the independent and the load as the dependent variable. This is attested to by all experimental values, beyond c_1 or c_2, which are recorded in Fig. 51.

A screw machine was purposely chosen for the test. In such a machine, when the switch is pulled, everything comes to a dead stop giving all the time required for the simultaneous load and deformation readings. In this test procedure lies one disadvantage. The peak of the curve for $l/i = 100$ may have been reached

but not recorded. This theoretical peak of $\frac{P}{A} = \frac{\pi^2 E}{(l/i)^2}$ is represented by a cross. The advantage of this test procedure, however, is that all experimental values recorded definitely represent equilibrium conditions. Such experimental values up to the first break in the curve we have, under item 3, defined as representing cases of elastic stability. The experimental values beyond the second break in the curve—beyond c_1 or c_2—are defined as representing *ductile equilibrium.* If the slenderness ratio of the column is l_1/i—and the column curve for $l/i = 100$ in. (Fig. 51) comes very close to representing such a column—the column passes suddenly from a state of elastic instability (see definition, page 92) into a state of ductile equilibrium. As appears evident from Fig. 53*f*, the value of P_3, representing a state of ductile equilibrium, is inevitably less than the value of P_1, which represents elastic instability or the upper limit of elastic stability for columns of a length l in which $0 < l < l_1$.

We have defined P_1 by the equation $\frac{P_1}{A} = \frac{\pi^2 E}{(l_1/i)^2}$ in which $\frac{P_1}{A}$ represents proportional limit stress. We would like to define P_2 by the equation $\frac{P_2}{A} = \frac{\pi^2 E}{(l_2/i)^2}$ in which $l_2 > l_1$. P_2 represents the elastic indifferent equilibrium value. This is exemplified by BC_1 or BC_2 in Fig. 51, as well as by the successive stages of varying stress distribution shown in Figs. 53*a*, *b*, and *c*. Figure 53*f* represents the limiting case of the series of stress distributions shown in Figs. 53*a*, *b*, *c*, and *d*. Thus, if P_2 is small the ductile-equilibrium condition is gradually approached. We may express this by saying that the elastic-indifferent-equilibrium condition is gradually eased into that of a ductile-equilibrium condition, and that the ductile-equilibrium condition differs numerically but slightly from the elastic indifferent equilibrium condition. This is exemplified by curves for $l/i = 198$ and $l/i = 306$ in Fig. 51, in which the horizontal line represents elastic indifferent equilibrium values and the curve beyond the horizontal line represents ductile-equilibrium values.

For the condition when $l = l_1$ for a material like mild steel with stress–strain properties like *ABC* (Fig. 52) the change from the condition of elastic instability to that of ductile equilibrium is

sudden and precipitous. The intermediate stress patterns represented by Figs. 53*b*, *c*, and *d* are not possible. The change from the one to the other condition is represented by Figs. 53*e* and *f*. It is also represented by the precipitous drop in value on the curve for $l/i = 100$ in Fig. 51. As P_2 approaches P_1 in value, the distance BC_1 or BC_2 (Fig. 51) decreases, the successive stress patterns shown by Figs. 53*a*, *b*, *c*, and *d* become less in number, and the transition from the elastic indifferent equilibrium condition to that of ductile equilibrium becomes more abrupt and is instantaneous when $P_2 = P_1$. Formula IX, presently to be developed, gives ductile equilibrium values P_3/A. It is shown graphically in Fig. 48 in which the Euler values represent P_1 in the short-column range and P_2 in the long-column range. The difference in ordinates between these two graphs represents $P_2 - P_3$. All points on the graphs beyond C_1 or C_2 (Fig. 51) represent ductile equilibrium values. P in formula IX represents the first ductile equilibrium value (P_3, Fig. 50*f*) obtained after buckling, when the deformations are considered the independent variable and the deformation, on buckling, remains unchanged. If the deformation continues, other ductile equilibrium values, smaller ones, are obtained. This is indicated in Fig. 53*g*, and by, say, successive values in the load-deformation curve for $l/i = 100$ in Fig. 51 beyond the second break in the curve.

6. Elastic Instability. The value $\dfrac{P_1}{A} = \dfrac{\pi^2 E}{(l_1/i)^2}$ which is constant (see formula V) for all l/i ratios, from $l/i = 0$ to $l/i = l_1/i$, represents *elastic instability*. A centrally loaded pin-ended column of a length $l < l_1$ reaches the value of s_1 and then suddenly buckles. The load-carrying capacity of such a column changes suddenly, on buckling, from P_1 to P_3. Figure 53*e* represents the stress distribution over a cross section of the column just before buckling, and Fig. 53*f* or *g* represents the stress distribution immediately after buckling. The two crosshatched rectangles in Fig. 53*f* or *g* represent the moment $P_3\Delta$. Since Δ is a function of l^2 it is largest for $l = l_1$. As l decreases from l_1 then Δ decreases; therefore $P_3\Delta$ decreases, and thus P_3 increases and $P_1 - P_3$, or the ordinate between formula IX and formula V (Fig. 48) becomes progressively smaller for values of l/i which are progressively less than l_1/i. Qualitatively this argument may be explained by means of Figs. 53*f* and *g*. Should Fig. 53*g* represent the ductile-equilibrium stress distribution for a column of a length l_1, then Fig. 53*f*, with a

larger P_3, would represent the ductile equilibrium stress distribution for a column of a length less than l_1.

In the discussion of ductile equilibrium we pointed out that the slenderness ratio l_1/i represents the limiting case for which the elastic indifferent equilibrium state, represented by the BC_1 or BC_2 (Fig. 51) approaches zero, and for which the precipitous drop in load-carrying capacity expressed as $P_2 - P_3$ reaches a maximum. In this chapter we have just argued that the precipitous drop in load-carrying capacity in the elastic instability range also reaches a maximum for the condition $l/i = l_1/i$. Thus $P_2 - P_3$ in the long-column range or $P_1 - P_3$ in the short-column range is a maximum for $l/i = l_1/i$. This is illustrated by the curves in Fig. 48 in which the maximum difference between the ordinates of these curves is found when $l/i = l_1/i$.

7. Ductile Instability. For many decades structural steel, economically the most important of structural materials, also dominated to a large extent theoretical and testing considerations. Mild steel exemplifies both nearly perfect elasticity and nearly perfect ductility. But in recent years the light metals have begun to challenge the position of dominance so long held by mild steel. In airplane construction the light metals apparently have superseded steel. Light metals are nearly perfectly elastic within a limited range (OF Fig. 52). The ductile behavior of light metals, however, is represented by a gradually varying stress–strain curve (F.H. Fig. 52) instead of an abruptly varying stress–strain curve (point B, Fig. 52).

Let us suppose an aluminum-alloy column centrally loaded to a value of P_2, which gives a value of P_2/A as represented by point G in Fig. 52 and an average load distribution P_2/A as represented by Fig. 53*c*. The modulus of elasticity, corresponding to the stress at point G in Fig. 52, is represented by the tangent to the stress–strain curve at G and is labeled E'. When simultaneous values of E', l/i, and P_2/A are reached, so as to satisfy the equation,

$$\frac{P_2}{A} = \frac{\pi^2 E'}{(l/i)^2} \qquad \text{Equation r}$$

we have a condition defined as *ductile instability*.

Equation (r) is called the tangent-modulus formula. In Fig. 50 the values of E' are determined by drawing tangents to the stress–strain curve as indicated in the figure. The simultaneous value

of these E''s and their corresponding P/A's are substituted in equation (r) which equation is then solved for l/i. The black dots and the solid black line for all values of $0 < l/i < l_1/i$ indicate the theoretical tangent-modulus curve. The circles indicate experimental values.

One consideration which is vital in the evaluation of formula IX, presently to be developed, is the relative difference in load-carrying capacity of columns represented by the elastic-instability and the ductile-equilibrium condition on the one hand and by ductile instability and ductile equilibrium on the other.

In mild-steel curve OBC (Fig. 52) the elastic instability, in terms of successive stress patterns, is represented by Figs. 53*e* and *f*.

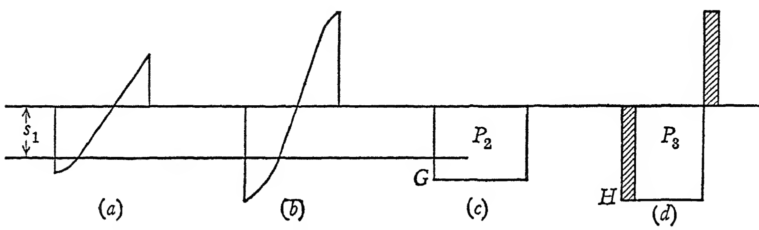


FIG. 56. Changes in Stress Distribution as an Aluminum-Alloy Column Buckles.

These are qualitative sketches, but it appears self-evident that, if P_1 represents the elastic-instability load and P_3 a ductile-equilibrium load, then P_3 is very materially smaller than P_1. We repeat that in this discussion we consider deformation the independent, and load the dependent, variable.

The phenomenon of passing from the condition of ductile instability to that of ductile equilibrium in columns of light-metal alloys differs markedly from the same phenomenon in mild-steel columns. When light-metal columns in the long-column range buckle, that is columns possessed of stress–strain properties as represented by curve $ODFGH$ (Fig. 52), the successive stress pattern will vary in a manner as shown by Figs. 56*a* and *b* until a limiting condition represented by Fig. 56*d* is reached. Such columns in the short-column range, however, buckle when we obtain the simultaneous values of E', l/i and P_2/A in equation (r), $\dfrac{P_2}{A} = \dfrac{\pi^2 E'}{(l/i)^2}$. This condition is represented by point G in Fig. 52, the stress pattern for which is represented by Fig. 56*c*. Upon

buckling, a stress pattern as represented by Fig. 56*d* is suddenly developed. From Fig. 52, however, it is obvious that, upon buckling, stresses are still capable of increasing from values represented by point G to those represented by point H. Thus the stress patterns, Figs. 56*c* and *d*, qualitatively resemble those for mild steel, Figs. 53*g* and *e*, but differ quantitatively to a marked degree. In mild steel the limiting stress shown in Fig. 53*e* is unable to rise above the value of s_1, whereas the limiting stresses in aluminum alloy are capable of increasing, after buckling, from those represented by point G in Fig. 56*c* to those represented by point H in Fig. 56*d*. Thus, although P_3 representing ductile equilibrium will be smaller than P_2 representing ductile instability, the drop in value, after buckling, from P_2 to P_3, in passing from one stage to another, will be less pronounced in aluminum-alloy columns than in columns of mild steel. These conclusions are conspicuously supported by experimental evidence. This evidence is presented in connection with the discussion of Fig. 60, page 104.

The stress patterns of Figs. 53*f* and 56*d*, representing ductile equilibrium of a buckled column of light metal, play a crucial part in the development of formula IX. When first presented in reference 1*j*, it was severely criticized as being insufficiently substantiated. This criticism is quite understandable. I concede that Fig. 56*d* is in the nature of an hypothesis, or an assumption. My support for this assumption lies in my experience with hundreds of column tests, personally conducted and personally very closely observed. Possibly I may liken myself to an erection foreman who in one look evaluates loads, bearing values, strength of timbers, and decides how to proceed correctly in an underpinning job. If I, with the aid of textbooks, handbooks, and slide rule, would undertake the same job, I would first require a great deal of time and probably, in good civil-engineering fashion, my results would turn out to be thousands per cent "on the safe side"—not to speak of the costly side. I readily concede that the erection foreman's one glance, even though he cannot rationalize it, would represent superior theory to that of my own. In column behavior during laboratory tests, I can often sense stresses without aid of instruments. By no means, however, will I offer this opinion as acceptable scientific evidence. I confess that I have clamped strain gages on both sides of a buckled column and found them registering strains of the same order of magnitude. Somewhat subconsciously I must have thought the evidence of little impor-

tance, since I did not record it and thus could not publish it. The acceptability of the assumption will ultimately be passed upon when the fate of formula IX is finally decided.

By way of summary Figs. 57 and 58 illustrate graphically, for

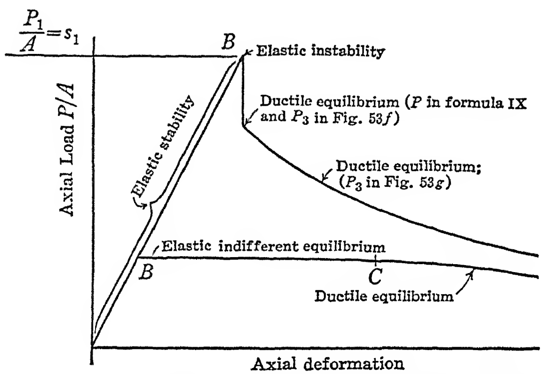


FIG. 57. Illustration of Definitions Used in Text. This figure is synthesized from Fig. 51.

mild steel, the concepts defined as critical column length, elastic stability, elastic indifferent equilibrium, ductile equilibrium, and elastic instability. For light metals the concepts defined as elastic indifferent equilibrium in the long-column range and as ductile

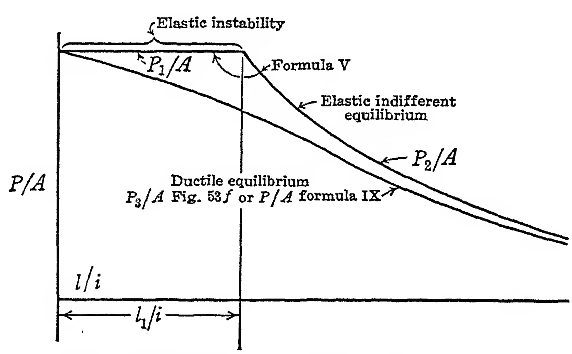


FIG. 58. Illustration of Definitions Used in the Text.

instability in the short- and intermediate-column ranges are illustrated by the experimental values shown as circles in Figs. 50 and 61. The concept defined as ductile equilibrium is shown by the graph for formula IX in these same figures.

DUCTILE-EQUILIBRIUM COLUMN FORMULA

The next five pages are a verbatim quotation from reference 1*j*.

An eccentrically loaded column assumes a shape, roughly, as indicated in Fig. 59*a*. The eccentricity e is augmented by an eccentricity Δ, the Δ resulting from the curvature of the column. *If we know the values for e and Δ, and also know the stress distribution over the cross-section area of the column, we can solve for the load P.* As to Δ, we know that it is a function of the length and of the curvature of the column. Unfortunately, however, we know nothing definite about the curvature of a column composed of varying modulus material and stressed far beyond the proportional limit throughout its length. But we do know from observation that the curvature does not assume the shape of two straight lines. We also know that a second degree equation can generally be fitted within close approximation to most any flat curve of single curvature. We therefore propose to assume that we may express Δ by the function $\Delta = kl^2$. *It should be realized that the constant k, in turn, is a function of the eccentricity e.*

As to the stress distribution over the cross-sectional area at the midsection of the column, we know that the compressive stress on the concave side of the column equals the upper yield point stress, here called s_2. For example, s_2 is 53,000 lb per sq in. for 24 ST, and 80,000 lb per sq in. for R 303 T, or 75 ST alloy. Further, we know, also from direct strain-gage measurements, that the tensile stresses on the convex side of the column, when the column supports its capacity load and when the tension stress–strain curve is the same as the compression stress–strain curve, likewise amount to this same stress s_2.

By upper yield point stress s_2 I mean an upper limit stress in which the stress–strain curve becomes nearly horizontal. As may be seen from Fig. 46 and from numerous stress–strain curves for aluminum and magnesium alloys in reference 1*i*, the compression stress–strain curve never assumes a zero slope. After a certain strain value is reached it goes on rising, apparently indefinitely. In this connection it must be remembered that the stress–strain curves which I have presented, and which are generally presented, are not true stress–strain curves. When a strain of the order of magnitude of 0.010 is reached, the cross-section area becomes something quite different from what it was when loading commenced. A true stress–strain curve, based on a stress determined by the

area which varies as a function of the load, rather than computed on the basis of original area, would thus have a much smaller slope. The conventional yield stress, defined by the 0.2% offset, is purely arbitrary, has no logical foundation, and thus cannot be used in any rational philosophy. For mild steel the upper yield stress s_2 is clearly defined.

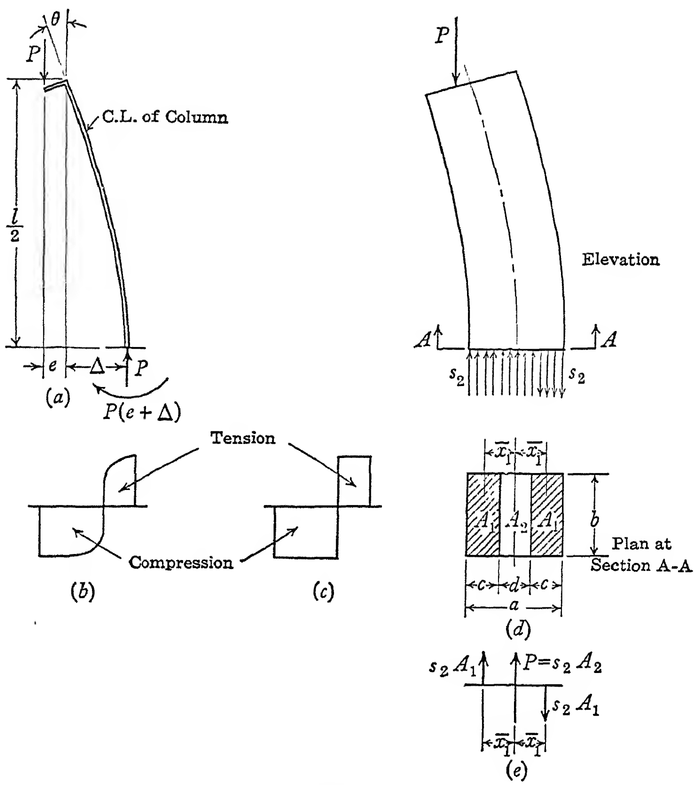


FIG. 59.

From what we know of elementary limit design (reference 1*b*) and residual stress analysis, the final stress distribution at the instant of failure, in case of identical tension and compression stress–strain curves, qualitatively looks something like Fig. 59*b*. We also know that if we simplify this sketch by giving it sharp corners and making it look like Fig. 59*c*, we introduce only a very minor error. The arrows, shown in Fig. 59*d* as acting on the cross section of the column, convey the same idea as that shown in

Fig. 59*c*. The stresses shown on Fig. 59*d* may be resolved into a Force $P = s_2A_2$ and a couple $s_2A_1(2\bar{x}_1) = P(e + \Delta)$ as shown in Fig. 59*e*.

As to the eccentricity e I have already dwelt at great length on the subject to show that any column analysis based on eccentricity has no practical value, since, in practice, e can only be definitely ascertained at the end of a complete analysis. It is a variable, in fact, it generally changes its sign. It is intangible, and slippery as an eel's tail. Much vaunted "engineering judgment" notwithstanding, I believe that anyone who assumes, or any committee who prescribes, a definite value for ec/i^2 is making a stab in the dark, in the pitch dark.

Any eccentric column analysis, thus, holds only academic, even though great, interest. This statement of course applies with equal force to our present analysis. By suitable laboratory control we may design our test equipment for an initial value of e which will hold substantially constant throughout the loading range. As the column is loaded the tangent to the elastic curve at the top of the column deflects; and if e represents the initial eccentricity, then the eccentricity which prevails at the end of the loading process should be written: $e \times \cos\theta$. (See Fig. 59*a*.) This variation in the value of e is a minor issue. However, when we ignore it, as we propose to do, we do not entirely neglect it, since the variation in e may also be assumed as a function of kl^2. Thus this variation in e will largely be included in the evaluation of the constant k.

On the basis of the assumptions here set forth, we may proceed with a mathematical analysis. Writing the equilibrium equation for Fig. 59, assuming a column of rectangular cross section, we obtain:

$$2s_2A_1\bar{x}_1 = P(e + \Delta) = P(e + kl^2)$$

$$s_2A_2 = P$$

$$A_2 = A - 2A_1 = ab - 2bc$$

$$2\bar{x}_1 = a - c$$

$$s_2(bc)(a - c) = s_2(ab - 2bc)(e + \Delta)$$

$$c^2 - [a + 2(e + \Delta)]c + a(e + \Delta) = 0$$

$$c = \frac{a + 2(e + \Delta) - \sqrt{a^2 + 4(e + \Delta)^2}}{2}$$

$$d = a - 2c = \sqrt{a^2 + 4(e + \Delta)^2} - 2(e + \Delta)$$

$$\frac{P}{A} = \frac{s_2 bd}{ab} = \frac{s_2 d}{a}$$

$$= \frac{2s_2}{a}\left[\sqrt{\frac{a^2}{4} + (e + kl^2)^2} - (e + kl^2)\right] \qquad \text{Formula VII}$$

Solving formula VII for k and substituting particular values for P and l, say $P = P'$ and $l = l'$, we obtain

$$k = \frac{\dfrac{a}{4}\left(\dfrac{As_2}{P'} - \dfrac{P'}{As_2}\right) - e}{(l')^2} \qquad \text{Equation s}$$

s_2 and a being constants, we may determine k by substituting the experimentally determined simultaneous values for P'/A and l' in equation (s), substitute this value for k in formula VII, and thus obtain a continuous expression of P/A as a function of l. This is not basically different from our usual procedure. In using Euler's equation, for example, we need a value for E. This value of E must be experimentally determined, and we may find this value from direct stress–strain measurements. Or, with the perfected method of testing pin-ended columns which we have described (references 1*e* and 1*i*), we very well might determine this value of E from only one test of a single pin-ended column, and use it to determine the buckling load of similar columns of varying lengths.

This latter procedure would result in establishing experimentally one constant for pin-ended columns loaded with zero eccentricity. In our case we have to establish experimentally the constant k for every series of tests, when each series has a different eccentricity.

I have stated that I hold any eccentric loading formula as being of only academic interest. But this is not true of the concentrically loaded pin-ended column. This, to my view, constitutes the essential criterion of value in the entire field of column behavior (references 1*e* and 1*f*), I am at present inclined to the opinion that this is nearly as true for columns stressed beyond the proportional limit as it is for those stressed within the elastic limit.

If we give e the value of zero in formula VII and in equation (s) we obtain

$$\frac{P}{A} = \frac{2s_2}{a}\left[\sqrt{\frac{a^2}{4} + (kl^2)^2} - kl^2\right] \qquad \text{Equation t}$$

also,

$$k = \frac{\frac{a}{4}\left(\frac{As_2}{P'} - \frac{P'}{As_2}\right)}{(l')^2} \qquad \text{Equation u}$$

If, in equation (t), we substitute equation (u) for k, we find, luckily for us, that the constant a cancels, and we obtain

$$\frac{P}{A} = 2s_2\left\{\sqrt{0.25 + \left[\frac{1}{4(l')^2}\left(\frac{As_2}{P'} - \frac{P'}{As_2}\right)\right]^2 l^4} - \frac{1}{4(l')^2}\left(\frac{As_2}{P'} - \frac{P'}{As_2}\right)l^2\right\} \qquad \text{Equation v}$$

The cancelling of the constant a does not mean that equation (v) becomes general. It is restricted, since it is based on the analysis of a rectangular section. It remains so restricted even after we cancel the constant a. However, it is believed to be quite rigorously applicable to Z and to hat stringers in aero design, and to most sections encountered in civil engineering practice.

Equation (v) is further restricted to the short column range and to columns made of materials with substantially identical tension and compression stress–strain curves. In the long column range we have the formulae in reference 1e to guide us. The transition value, which distinguishes between the short and the long column range is the l/i which corresponds to the proportional limit stress. In terms of graphs this means that the Euler curve and the curve of equation w, presently to be developed, should have one point in common, corresponding to the proportional limit stress s_1. If we substitute s_1, the proportional limit stress, for P'/A in equation (v), we obtain the *Limit Strength Column Formula for Material of Variable Modulus*

$$\frac{P}{A} = 2s_2\left\{\sqrt{0.25 + \left[\frac{1}{4l_1{}^2}\left(\frac{s_2}{s_1} - \frac{s_1}{s_2}\right)\right]^2 l^4} - \frac{1}{4l_1{}^2}\left(\frac{s_2}{s_1} - \frac{s_1}{s_2}\right)l^2\right\}$$

Equation w

The last quoted paragraph and particularly the last sentence in it, "If we substitute s_1, the proportional limit stress, for P'/A in equation (v) we obtain etc. . . ." involves an inconsistency and calls for revision.

I now believe that equation (v) is not restricted to the short-column range, and, further, it seems inconsistent to substitute an elastic-instability expression, $s_1 = P_1/A$, in an avowedly ductile-equilibrium equation. We have dwelt on the fact (page 92) that the relative difference between the elastic indifferent equilibrium load and the ductile-equilibrium load $P_2 - P_3$ decreases as the length increases beyond the critical length l_1. For very long columns this value becomes substantially zero. This is illustrated by the curve for $l/i = 198$ (Fig. 51) which, beyond C_2, is seen to ease very gradually from the elastic indifferent equilibrium state into that of ductile equilibrium.

Equation (w) may thus be said to be correct, but its interpretation should be different from that given to it in reference 1*j*. Instead of assigning to s_1 and l_1 values equal to the proportional-limit stress and the critical length we replace s_1 and l_1 in equation (w) by s' and l' where s' and l' represent simultaneous Euler values in the long-column range. Equation (w) then becomes

$$\frac{P}{A} = 2s_2 \left\{ \sqrt{0.25 + \left[\frac{1}{4(l')^2} \left(\frac{s_2}{s'} - \frac{s'}{s_2} \right) \right]^2 l^4} - \frac{1}{4(l')^2} \left(\frac{s_2}{s'} - \frac{s'}{s_2} \right) l^2 \right\}$$

where

$$s' = \frac{\pi^2 E i^2}{(l')^2} \qquad \text{or} \qquad (l')^2 = \frac{\pi^2 E i^2}{s'}$$

and by substitution we obtain

$$\frac{P}{A} = 2s_2 \left\{ \sqrt{0.25 + \left[\frac{l^2}{4\pi^2 E i^2} \left(s_2 - \frac{(s')^2}{s_2} \right) \right]^2} - \frac{l^2}{4\pi^2 E i^2} \left(s_2 - \frac{(s')^2}{s_2} \right) \right\}$$

or

$$\frac{P}{A} = 2s_2 \left[\sqrt{0.25 + \left(\frac{s_2^2 - (s')^2}{4s_{cr}s_2} \right)^2} - \frac{s_2^2 - (s')^2}{4s_2 s_{cr}} \right] \qquad \text{Formula VIII}$$

This is the same as saying that, in the very long-column range, the Euler values and the experimental values for ductile equilibrium substantially agree. Should we select, for example, $s' = 12{,}000$ psi, this would mean that for this value formula VIII

and the Euler formula have one point, the point corresponding to $s' = 12{,}000$, in common. We have stated our reasons (page 102) why the selection of s' as equal to s_1, as was done in reference 1j, seems unjustifiable. Furthermore, in the discussion of ductile equilibrium (page 91) we have explained how the values for elastic indifferent equilibrium (Euler values) and ductile equilibrium approach each other in the long-column range. The selection of a proper Euler value to substitute in formula VIII, we are unable to rationalize other than to say that it should be an Euler value in the long-column range. One of the graphs in Fig. 61 is the graph for formula VIII when the value of 12,000 psi is substituted for s'. Substituting in formula VIII, $s' = 0$ instead of $s' = 12{,}000$ would signify that formula VIII and the Euler formula, for the value of l/i equal to infinity and of $s' = 0$, have one point in common. Such a substitution simplifies formula VIII to read

$$\frac{P}{A} = s_2\left[\sqrt{1 + \left(\frac{s_2}{2s_{cr}}\right)^2} - \frac{s_2}{2s_{cr}}\right] \qquad \text{Formula IX}$$

in which $s_{cr} = \dfrac{\pi^2 E}{(l/i)^2}$, where E is the initial modulus and s_2 is yield stress.

In Fig. 61 one graph represents formula VIII plotted for a value of $s' = 12{,}000$ while the other one represents formula IX. Since a measure of arbitrariness is involved in the selection of a value for s' other than the stipulation that it should represent an Euler value in the long-column range, and since, furthermore, the two equations present graphs which differ by only very small amounts, we pronounce formula IX as quite reasonable and, because of its simplicity, as preferable to formula VIII.

In the expression for s_{cr} in formula IX, l may be written as $n\,l$ to express end restraints. In legs of transmission towers, in their diagonal bracing or in their horizontal cross strut, n should be taken as unity. This, in my judgment, should also be done in designs of airplane stringers. The graph for formula IX, with reference to pin-ended steel columns, is shown in Fig. 48; and for pin-ended 24 ST and 75 ST aluminum-alloy columns it is shown in Fig. 60 and Fig. 61. In neither Fig. 48, Fig. 60 nor Fig. 61 does the graph for formula IX fit the experimental values. This could not be expected because the experimental values represent elastic

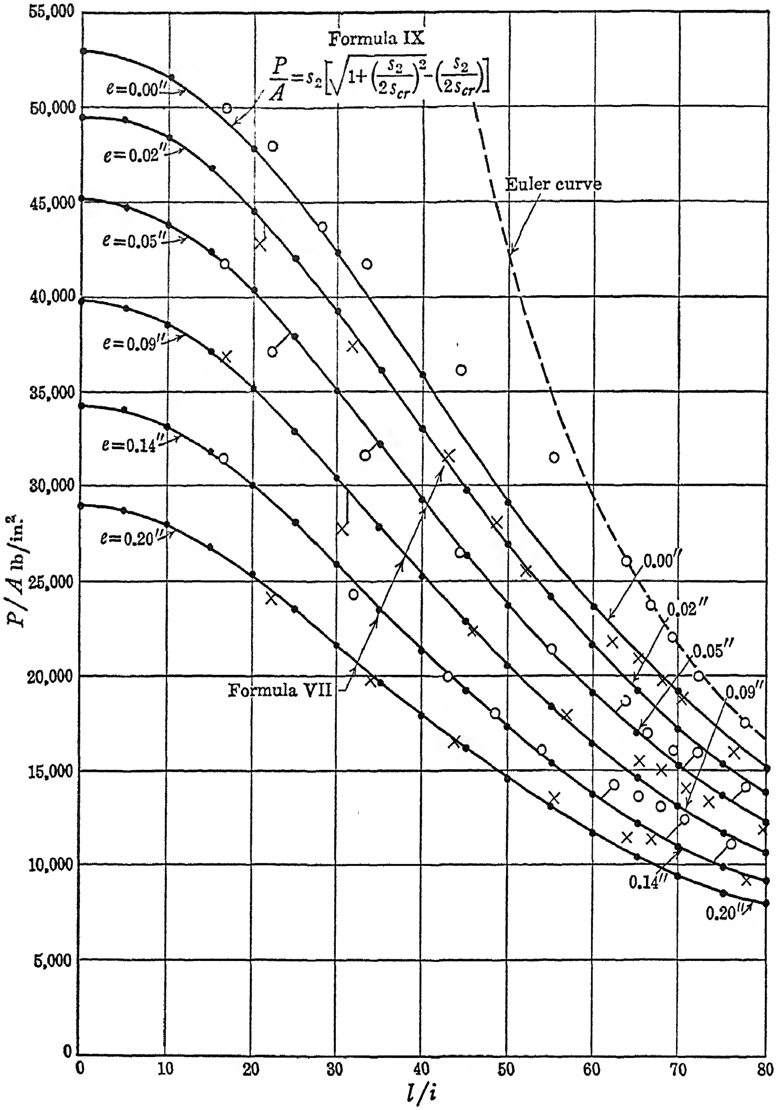


FIG. 60. Limit Loads of Eccentrically Loaded Columns—Experimental Verification of Formula VII. Columns were ⅝″ square 24 ST aluminum-alloy extrusion.

or ductile instability or elastic indifferent equilibrium, whereas formula IX represents ductile equilibrium. In this, I believe, lies its special merit.

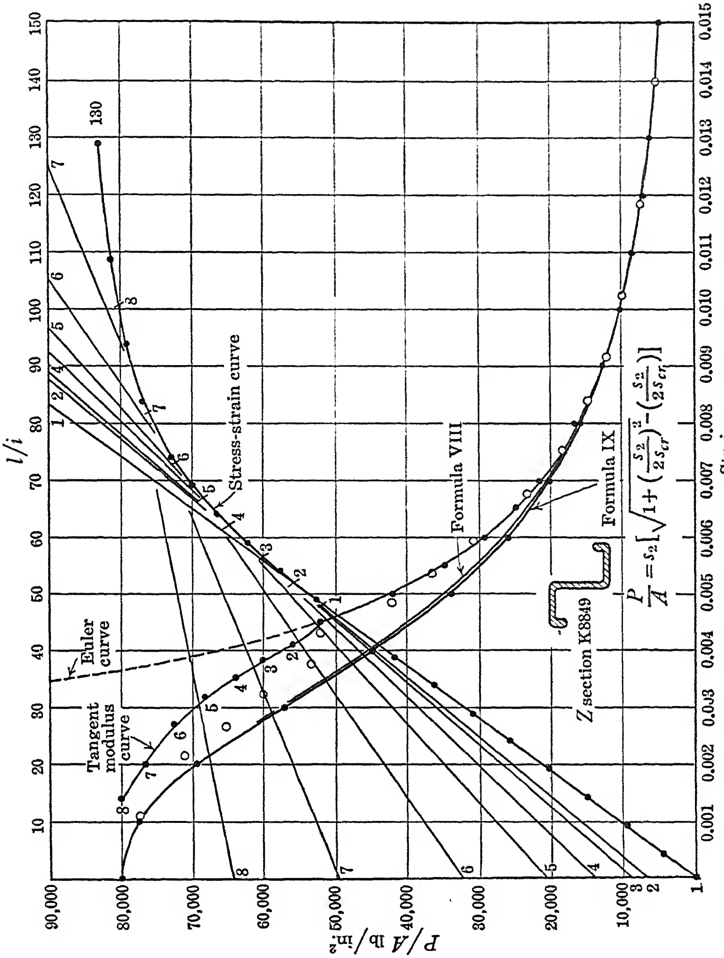


FIG. 61. 75 ST Aluminum Z Stringer Concentrically Loaded Column Tests.

Evaluation of Formula IX. Figures 46 and 60 are two graphical records of one and the same series of tests. Aluminum-alloy columns ⅝ in. square were tested with predetermined eccentricities. A strain gage was clamped on the concave side of the specimens. When the reading of this gage corresponded to a similar reading for $s = 27{,}000$ psi obtained from the stress–strain curve on Fig. 46, which was determined from a concentric com-

pression test, it was concluded that a stress of 27,000 psi in the column was reached. Simultaneously with this gage reading the value of the load supported by the column was recorded. These loads divided by area gave the P/A which is recorded in Fig. 46, for a number of columns of different length as well as for a number of columns tested with different eccentricities. The black lines are graphs obtained by substituting 27,000 for s_1 and 10,640,000 psi for E in formula III. The graphs representing formula III were drawn without reference to the experimental values. After the readings for Fig. 46 were taken for any individual test, the test was continued until the maximum load supported by the column was reached and recorded. These experimental maximum load values are recorded in Fig. 60.

Plotting of the graphs for formula VII, shown in Fig. 60, involves the determination of values for the yield stress s_2 and for the constant k. In mild steel the yield stress is clearly defined (see Fig. 48). In light-metal alloys the determination of the yield stress s_2 is somewhat arbitrary and involves the exercise of one's judgment. The equivalent of Fig. 60, as first published in reference 1j, was based on a yield stress $s_2 = 50{,}000$ psi. Figure 60 as published in this book is based on a yield stress $s_2 = 53{,}000$ psi.

The constant k may be determined rationally only for the case when $e = 0$. Then, according to equation (u),

$$k = \frac{\dfrac{a}{4}\left(\dfrac{As_2}{P'} - \dfrac{P'}{As_2}\right)}{(l')^2} = \frac{\dfrac{a}{4}\left(\dfrac{s_2}{s'} - \dfrac{s'}{s_2}\right)}{(l')^2}$$

in which $(l')^2 = \pi^2 Ei^2/s'$ and s' and l' represent simultaneous values in the very long-column range. In formula IX, which is supposed to have one point in common with the Euler formula at infinity, $s' = 0$, and equation (u) becomes

$$k = \frac{a\left[s_2 - \dfrac{(s')^2}{s_2}\right]}{4\pi^2 Ei^2} = \frac{as_2}{4\pi^2 Ei^2} \qquad \text{Equation x}$$

If we substitute for a the dimension of the test specimen $a = \frac{5}{8}$ in. and for s_2 the value $s_2 = 53{,}000$ psi, then $k = 0.00242$. In determining the constant k by means of substituting, in equation (s), simultaneous experimental values for P', s_2, e and l' it was found that k varied linearly as a function of e. This agrees with the

assumption expressed in the last sentence of the first paragraph of this chapter. The different values of k, for all curves shown in Fig. 60, are defined by the equation:

$$k = 0.00242 + 0.0083e$$

It may be argued that by substituting $(l')^2 = \pi^2 Ei^2/s'$ in equation (s) we obtain

$$k = \frac{\frac{a}{4}\left[s_2 - \frac{(s')^2}{s_2}\right] - es'}{\pi^2 Ei^2} \qquad \text{Equation y}$$

and that equation (y) reduces to the same expression as does equation (u), namely equation (x), when we let $s' = 0$. This then would mean that k is a constant and independent of e. Such an interpretation placed upon our mathematics, we believe, is unwarranted. It is conceded that the effect of e on column strength decreases as the column length increases and would be nil for columns of infinite length. To hold that an eccentric-column formula may ever be given a value in common with a concentric-column formula, for columns of normal proportions, is against reason. That a ductile-equilibrium formula for centrally loaded columns, on the other hand, may be assumed to have a point in common with the Euler value in the long-column range is discussed at length on page 91 and is attested to by the graphs for formula IX and formula VIII in Figs. 48 and 61, respectively.

Formula III is the result of the solution of a quadratic equation which presents Euler's formula, formula V, as a limiting case when $e = 0$, and gives the value of P/A as s_1 in the short-column range and as $\frac{\pi^2 E}{(l/i)^2}$ in the long-column range. Formula VII also results from the solution of a quadratic equation, one closely similar to the one involved in the derivation of formula III. In general outline, the graphs for formula III (Fig. 46) and those for formula VII (Fig. 60) are also similar. Formulas III and V, which are theoretically valid, may thus be interpreted as lending a measure of support to formulas VII and IX. The distinction between the two sets of formulas is that formulas III and V are predicated on the assumption that the proportional-limit stress represents the criterion of column strength, whereas in the derivation of formulas VII and IX ductile equilibrium is considered to be the criterion of strength. By calling formulas III and V "theo-

retically valid" we meant that the logic on which they are based seems unimpeachable. They seem invalid as engineering-design formulas because the assumptions underlying this logic seem untenable.

Discrepancy between Experimental Values and Theoretical Curves. The mathematical analysis incident to the derivation of formula IX tells only a part of the story. The graphs and experimental results shown in Figs. 48 and 60 tell another part. Still another and a very important part of the same story, important in the evaluation of formula IX, remains to be told but must be told in words. I do not know that we have a good definition of the word buckling. What most of us understand this word to mean, I believe, is the development of a sudden side kick in a column under load, and a corresponding decrease in the load-carrying capacity of the column. We have dwelt on this phenomenon earlier in this treatise and expressed it in the form, $P_1 - P_2$ or $P_2 - P_3$. We have stated the fact and explained that this quantity is largest for centrally loaded columns of a length l_1. It is a very striking phenomenon in the laboratory that the drop in load-carrying capacity of centrally loaded columns, upon buckling, decreases as the length of the column increases beyond the value of l_1, and decreases also as the length decreases from $l = l_1$ to $l = 0$. A similar phenomenon may be noted in testing eccentrically loaded columns. The stress pattern of eccentrically loaded columns ultimately approaches the pattern represented by Fig. 56*d*, but this pattern is approached through successive stages as shown in Figs. 56*a* and *b*, and not suddenly from the stage shown in Fig. 56*c* to that of Fig. 56*d*, as is the case with centrally loaded columns. The maximum load on an eccentrically loaded column is gradually reached and is likewise gradually decreased, if the deformation of the column continues. This phenomenon is most pronounced when the eccentricity is large. As the eccentricity decreases, as it approaches closely to the value $e = 0$, the column behavior corresponds ever more closely to that of a concentrically loaded column. The experimental values in Fig. 60, when $e = 0$, represent elastic indifferent equilibrium in the long-column range and ductile instability in the short-column range. The graphs for formula IX represent ductile equilibrium. For the column with zero eccentricity, $e = 0.00$ in., the recorded maximum test load represents more nearly elastic indifferent equilibrium or ductile instability than ductile equilibrium. This

accounts for the discrepancy between theoretical and experimental values for this curve as shown in Fig. 60.

Theoretical Values Are More Significant than Experimental Values. Satisfactory column analysis is seriously hindered by the difficulty of simulating actual service conditions in the laboratory. Emphasis in theory and tests has been directed towards the ideal pin-ended column, because, although never found in actual practice, it constitutes one ideal condition subject to theoretical as well as to empirical analysis. The matter of inevitable eccentricities is generally recognized. It is sometimes claimed that a column design formula cannot be rational unless it contains a term expressing eccentricity. This I believe to be inherently impossible. At least up to the present I have seen no indication pointing to a satisfactory design formula containing the term e, other than the ordinary eccentric-loading formula applicable to pillars. I have called the secant formula, formula IV, exact, and I hold formula III to be exact enough for all practical purposes. I believe I have substantiated the latter claim with experimental evidence (Fig. 46). But I hold both formulas III and IV to be futile as design formulas. The same goes for formula VII. Nevertheless, formulas III and VII perform a very useful function. They provide a family of curves the limiting case of which when $e = 0$, as in formula III, leads to the Euler formula (Fig. 48) or, if we substitute E' for E, to the tangent modulus formula (Fig. 46). In the case of formula VII, when $e = 0$, it leads to formula IX. All five curves given in Fig. 60 for columns with eccentricities other than zero, lend support to the one case that essentially interests us, the ductile-equilibrium formula for a column centrally loaded.

Formula IX, I believe, may be likened to formula I, $s = P/A$. I hold formula I to be a limit-design formula, because if it were considered as an elasticity formula we would have to take into account inevitable scratches, lack of alignment, wow, and so on, all of which would invalidate formula I. Similarly, inevitable wow, eccentricities, and other imperfections invalidate Euler's formula as a design formula. Minor eccentricities, wow, or other imperfections, however, would not affect ductile-equilibrium values. I claim for formula IX the following:

1. It is rational.
2. It represents a limiting case of a family of curves, all fairly well empirically verified.

3. Minor eccentricities, wow, or other imperfections do not affect ductile-equilibrium values, thus dispensing with the necessity of incorporating e in the formula.

4. It is applicable to steel and to light-metal columns alike; in fact, it is applicable to all columns composed of materials possessed of substantially identical tension and compression stress–strain curves.

5. It covers the entire range of values of slenderness ratios from $l/i = 0$ to $l/i = \infty$.

6. In its essential aspects it agrees with all empirical design formulas in present use.

7. It is very simple.

Of the 30 or more steel-column design formulas in current use, the one laying greatest claim to being rational is the Rankine formula: *

$$P/A = \frac{s}{1 + sl/ki}$$

If the Rankine formula is made tangent to the Euler curve at infinity, it assumes the form:

$$P/A = \frac{s}{1 + s/s_{cr}} \qquad \text{Modified Rankine formula}$$

This formula is appealing because it involves both the terms E and (l/i) which we know affect column strength profoundly. Its general shape, when plotted, is also of a type to appeal to the engineer's judgment. To the author's knowledge it is never used as a design formula because in the range of column lengths in most common use it gives values which are considered too low. It is thus quantitatively unacceptable even though qualitatively appealing.

We call attention to the modified Rankine formula in this instance because of its relation to the Rankine formula which is held in high esteem. Further, it may be noted that it is of the same general form as equation (w) in Fig. 62. Equation (w) is one of the old work horses in which we have gained confidence through constant use. Its only flaw is that the e in it is regarded as a constant, which in fact it never is.

* The term l/i seems to occur only in the modern version of the Rankine formula. In his "Manual of Applied Mechanics," 1870 (he died in 1872), Rankine proposes the term l/h where h is defined as "the least diameter." In neither his "Miscellaneous Scientific Papers" nor his "Civil Engineering" was I able to find any reference to l/i.

Equation (x) in Fig. 62 is derived by replacing the term $(e + kl)$ of formula VII by e. Equation (x) in Fig. 62 is just as sound as equation (w), provided we accept the limit stress pattern of Fig. 62*f* as reasonable.

It seems worthy of note that formula IX is of the same form

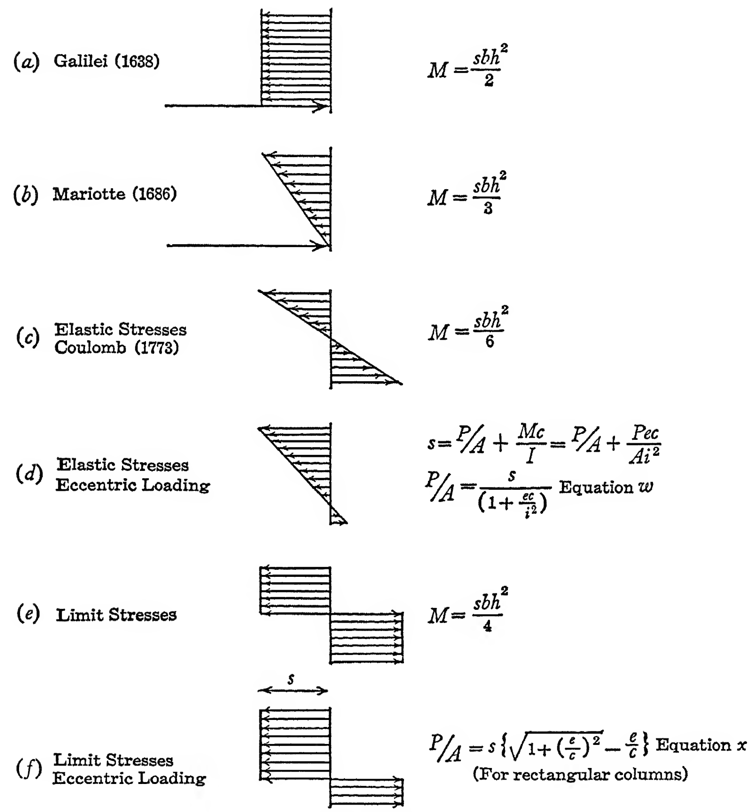


FIG. 62. Various assumed stress distributions in beams and columns.

as equation (x) in Fig. 62. What appears as a constant (e/c) in the latter appears as a variable $s/2s_{cr}$ in the former. I do not advance the similarity between formula IX and equation (x) as proof that formula IX is correct. However, since equation (x) is as rational as equation (w), such as it is, it seems to me that the analogy between formula IX and equation (x) lends support to the claim that formula IX is valid.

TWO-PANEL COLUMN FORMULA *

Continuous columns extending over two panels of unequal length, such as the cross bracing in transmission towers, are of common occurrence. If a is the shorter and b the longer of the

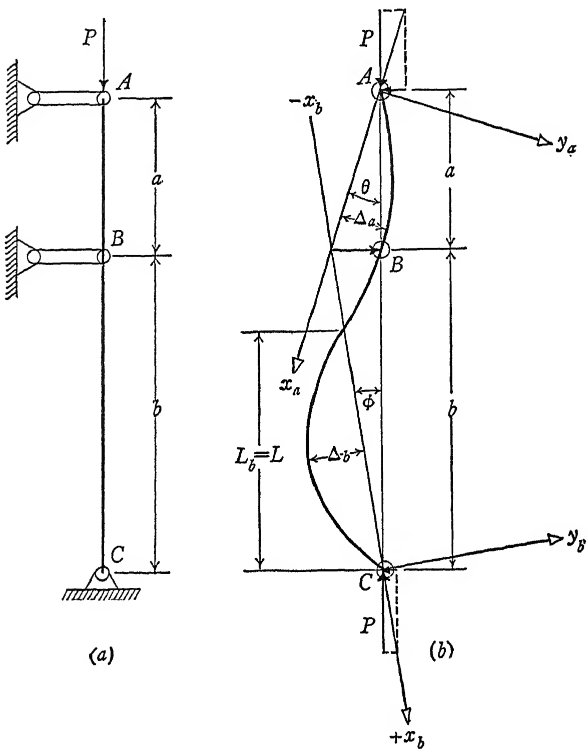


FIG. 63.

two lengths into which the continuous column is divided (see Figs. 63 and 65), then

$$\frac{\pi^2 E}{(a/i)^2} > \frac{P}{A} > \frac{\pi^2 E}{(b/i)^2}$$

* The two-panel column formula has been mathematically derived and experimentally checked by Mr. Evan A. Fradenburgh. This work was executed under the author's direction as part requirement for a master's degree in aeronautical engineering at the University of Michigan.

This may be written as

$$\frac{P}{A} = \frac{\pi^2 E}{(L/i)^2} = \frac{\pi^2 E}{(nb/i)^2} = \frac{C\pi^2 E}{(b/i)^2}$$

where L is the full arch of a sine curve column length and C is a fixity coefficient. For one limiting case, when $a/b = 1$, $C = 1$ and $P/A = \pi^2 E/(b/i)^2$. The other limiting case, when $a/b = 0$, represents a column hinged at one end and built in at the other end, for which $C = 2.046$ or $P/A = 2.046\pi^2 E/(b/i)^2$. The values of n and C for intermediate values of a/b are determined as follows:

Figure 63a represents a continuous pin-ended column with an intermediate pinned support, so that only vertical motion of the support is permitted. When the column buckles, it will assume the shape shown in Fig. 63b. There will be horizontal reactions as shown, so that the resultant forces at the ends A and C will not be collinear but will intersect at a point a small distance horizontally from B. The magnitudes of the resultant forces will be approximately equal to the applied force P.

For small deflections, the elastic curve of a pin-ended column is a sine curve about the line of action of the resultant end load. Therefore, if two co-ordinate systems are chosen as shown, the elastic curve from A to B is a sine curve about the x_a axis; the curve from C to B is a sine curve about the negative x_b axis. Let Δ_a, Δ_b, and L_a, L_b, be the amplitudes and half wave lengths of the two curves, respectively. The length L_b determines the load-carrying capacity of the column, since, by Euler's formula, $P_{cr} = \pi^2 EI/L^2$ where $L = L_b =$ critical column length. Thus the problem is to evaluate L_b in terms of the span lengths, a and b.

The equations of the two sine curves may be written:

$$y_a = \Delta_a \sin\left(\frac{\pi}{L_a} x_a\right) \qquad \text{Equation aa}$$

$$y_b = -\Delta_b \sin\left[\frac{\pi}{L_b}(-x_b)\right] = \Delta_b \sin\left(\frac{\pi}{L_b} x_b\right) \qquad \text{Equation bb}$$

It will be necessary to find the expressions for the slope and curvature of these curves, and thus the equations are differentiated.

$$\frac{dy_a}{dx_a} = \Delta_a \frac{\pi}{L_a} \cos\left(\frac{\pi}{L_a} x_a\right) \qquad \text{Equation cc}$$

$$\frac{dy_b}{dx_b} = \Delta_b \frac{\pi}{L_b} \cos\left(\frac{\pi}{L_b} x_b\right) \qquad \text{Equation dd}$$

and

$$\frac{d^2y_a}{dx_a^2} = -\Delta_a \frac{\pi^2}{L_a^2} \sin\left(\frac{\pi}{L_a} x_a\right) = -\frac{\pi^2}{L_a^2} y_a \qquad \text{Equation ee}$$

$$\frac{d^2y_b}{dx_b^2} = -\Delta_b \frac{\pi^2}{L_b^2} \sin\left(\frac{\pi}{L_b} x_b\right) = -\frac{\pi^2}{L_b^2} y_b \qquad \text{Equation ff}$$

If it is assumed that the angles θ and ϕ are small, so that $\tan\theta = \theta$, $\tan\phi = \phi$, and $\cos\theta = \cos\phi = 1$, the four boundary conditions are as follows:

First At $x_a = a, \quad y_a = a\theta = b\phi$

Second At $x_b = -b, \quad y_b = b\phi$

Then from equations aa and bb,

$$b\phi = \Delta_a \sin\left(\frac{\pi a}{L_a}\right) \qquad \text{Equation gg}$$

and

$$b\phi = -\Delta_b \sin\left(\frac{\pi b}{L_b}\right) \qquad \text{Equation hh}$$

Third At the intermediate support, or at $x_a = a$ and $x_b = -b$, the slope of the elastic curve is the same on both sides of the support. Since, however, the axes of the sine curves differ by an angle $(\theta + \phi)$, we have

$$\frac{dy_a}{dx_a} = \frac{dy_b}{dx_b} + (\theta + \phi) \quad \text{at that point.}$$

Then from equations cc and dd,

$$\Delta_a \frac{\pi}{L_a} \cos\left(\frac{\pi a}{L_a}\right) = \Delta_b \frac{\pi}{L_b} \cos\left(\frac{\pi b}{L_b}\right) + (\theta + \phi) \qquad \text{Equation ii}$$

Fourth At $x_a = a$ and $x_b = -b$, the curvature of the elastic curve is the same on both sides of the support, since the bending moment is the same immediately on both sides. Then, from equations ee and ff,

$$-\frac{\pi^2}{L_a^2} y_a = -\frac{\pi^2}{L_b^2} y_b \qquad \text{where} \quad y_a = y_b = b\phi$$

$$\frac{\pi^2}{L_a^2} = \frac{\pi^2}{L_b^2} \quad \text{or} \quad L_a = L_b$$

The subscripts may now be dropped, so that

$$L_a = L_b = L$$

From equations gg and hh,

$$\Delta_a = \frac{b\phi}{\sin\left(\dfrac{\pi a}{L}\right)} \quad \text{and} \quad \Delta_b = -\frac{b\phi}{\sin\left(\dfrac{\pi b}{L}\right)}$$

Substituting these values for Δ_a and Δ_b in equation ii gives

$$\frac{b\phi}{\sin\left(\dfrac{\pi a}{L}\right)} \frac{\pi}{L} \cos\left(\frac{\pi a}{L}\right) = -\frac{b\phi}{\sin\left(\dfrac{\pi b}{L}\right)} \frac{\pi}{L} \cos\left(\frac{\pi b}{L}\right) + (\theta + \phi)$$

$$\frac{\pi b\phi}{L} \cot\left(\frac{\pi a}{L}\right) = -\frac{\pi b\phi}{L} \cot\left(\frac{\pi b}{L}\right) + (\theta + \phi)$$

$$\frac{\pi b}{L}\left[\cot\left(\frac{\pi a}{L}\right) + \cot\left(\frac{\pi b}{L}\right)\right] = \frac{\theta + \phi}{\phi}$$

Now

$$\frac{\theta + \phi}{\phi} = 1 + \frac{\theta}{\phi} = 1 + \frac{b}{a} \qquad \text{since} \quad a\theta = b\phi$$

Thus, we have a final equation relating L with the dimensions a and b:

$$\frac{\pi b}{L}\left[\cot\left(\frac{\pi a}{L}\right) + \cot\left(\frac{\pi b}{L}\right)\right] = 1 + \frac{b}{a}$$

The solution of this trigonometric equation was found by a trial-and-error method for various ratios of a to b and is plotted

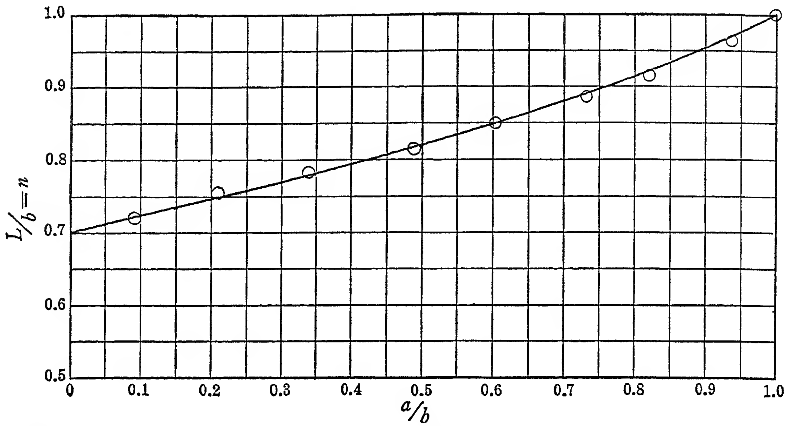


FIG. 64. Experimental values (circles) and theoretical values (graph) for coefficient n = L/b for two-panel columns.

in Fig. 64. The circles shown represent experimental points obtained with the testing machine (Fig. 65).

The critical load is found by Euler's formula:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

or

$$P_{cr} = \frac{1}{(L/b)^2}\frac{\pi^2 EI}{b^2} = \frac{1}{n^2}\frac{\pi^2 EI}{b^2} = C\frac{\pi^2 EI}{b^2} \qquad \text{Equation jj}$$

where

$$n = \frac{L}{b}, \qquad C = \frac{1}{n^2} = \frac{1}{(L/b)^2}$$

This coefficient C is plotted in Fig. 66, the circles representing experimental points. It is seen that the straight line given by the equation $C = 2 - a/b$ closely approximates the exact curve, with the error on the safe side. Substituting this value for C in equation jj results in

$$P_{cr} = \left(2 - \frac{a}{b}\right)\frac{\pi^2 EI}{b^2}$$

or

$$\frac{P}{A} = \left(2 - \frac{a}{b}\right)\frac{\pi^2 E}{(b/i)^2} \qquad \text{Formula X}$$

FIG. 65 (right). Fixture for testing two-panel columns.

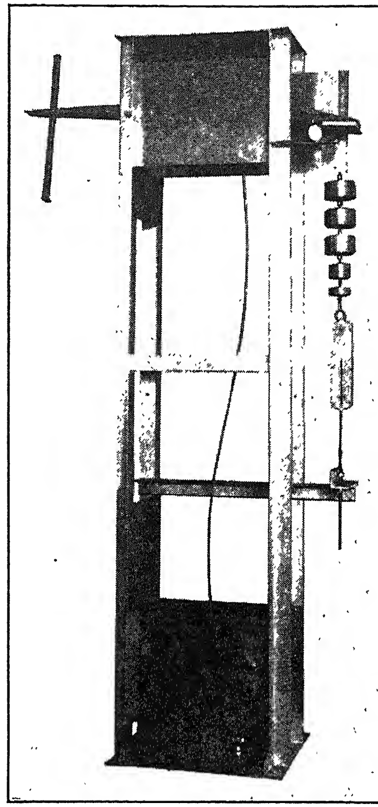
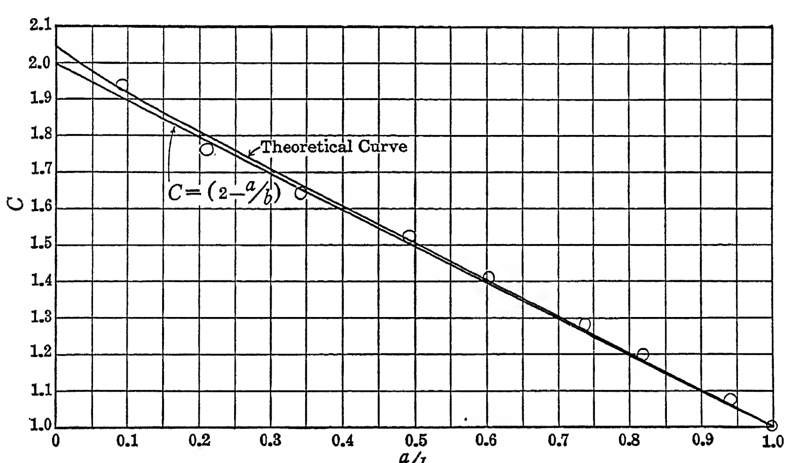


FIG. 66 (below). Fixity coefficient C for two-panel columns. Experimental values are shown by circles. The discrepancy between theoretical and approximate (C = 2 − a/b) values, is indicated by the two graphs.



TRANSMISSION TOWERS

The transverse shear in transmission towers is resisted principally by the cross bracing. Design may be based on one of three theories of strength.

1. *The elastic stress theory of strength* stipulates that at no point in a structure may the working stress exceed a certain proportion of either the tensile elastic-limit stress or of the buckling P/A of compression members. This philosophy, carried to its logical conclusion, would demand in the majority of cross bracings that the compression diagonal be made unduly heavy. The possibility of complete reversal of loading, not to speak of construction and other considerations, generally prescribes symmetry. It thus would follow that the tension diagonal be made equally as heavy as the compression diagonal.

2. The structural-engineering profession offers a convenient rule. *Cross bracing is designed on a* 50–50 *basis or on a* 100–0 *basis.* It is commonly assumed that in cross bracing the compression diagonal may be completely ignored. In making this assumption the truss is regarded as a statically determinate one. Thus complexities in theoretical analysis are reduced to a minimum, and the resulting design, in the vast majority of cases, will be considerably more economical than if the philosophy mentioned under item 1 were followed, and, last but not least, the error, and an error is obviously involved, will be "on the safe side." It is not contested that "erring on the safe side," and err we inevitably do, is commendable. However, considerations of weight–strength efficiency (aero design) and economy strongly urge the reduction of our inevitable errors to a minimum.

3. *The theory of limit design* assumes that the sustaining power of a column, after buckling, may be determined with reasonable accuracy. Once so determined it may be used to determine the limit strength of the structure of which it forms an integral part. Once we know the limit strength we also know the safe strength. The evidence supporting this assumption has already been presented. We propose to discuss this evidence but not until after first presenting the analysis of towers.

Analysis of Towers. Figure 67*a* represents a tower subject to a transverse load P applied at point A. Figures 67*b*, *c*, and *d* represent free-body sketches of the forces acting at points A, B, and D, respectively. Figures 67*e*, *f*, and *g* represent force polygons for

the afore-mentioned free-body sketches. The force polygons Figs. 67*e*, *f*, and *g* may be combined into a single force polygon as shown in either Fig. 68*a*, Fig. 68*b* or Fig. 68*c*. The members in

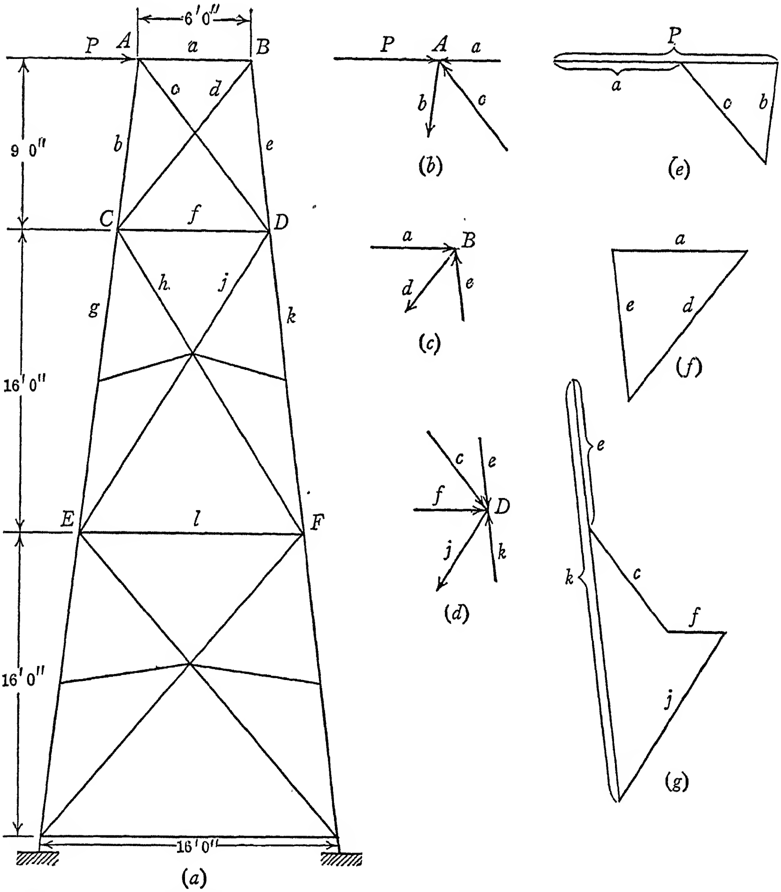


Fig. 67. Transmission Tower and Force Diagrams for Some of Its Joints.

the truss (Fig. 67*a*) as well as the corresponding forces in the force diagrams are represented by lower-case letters.

The laws of static equilibrium tell us a great deal, but so far they give us only qualitative results. They tell us that the transverse shear carried by the diagonals bears a definite ratio to the load *P*, regardless of how it is distributed between the two diagonals. The forces transmitted by the diagonals are shown by the

parallelogram *dcdc* in Fig. 68. An infinite number of solutions is possible. An infinite number of parallelograms *dcdc* may be drawn. This includes the possibility of the sides *d* and *c* of the parallelogram being equal, in which case it will be a rhombus, or of the force *c* being zero, in which case the parallelogram becomes a single line (Fig. 68*c*).

If the rule of ignoring the compression diagonal (mentioned under item 2 on page 118) be followed, then the truss is statically

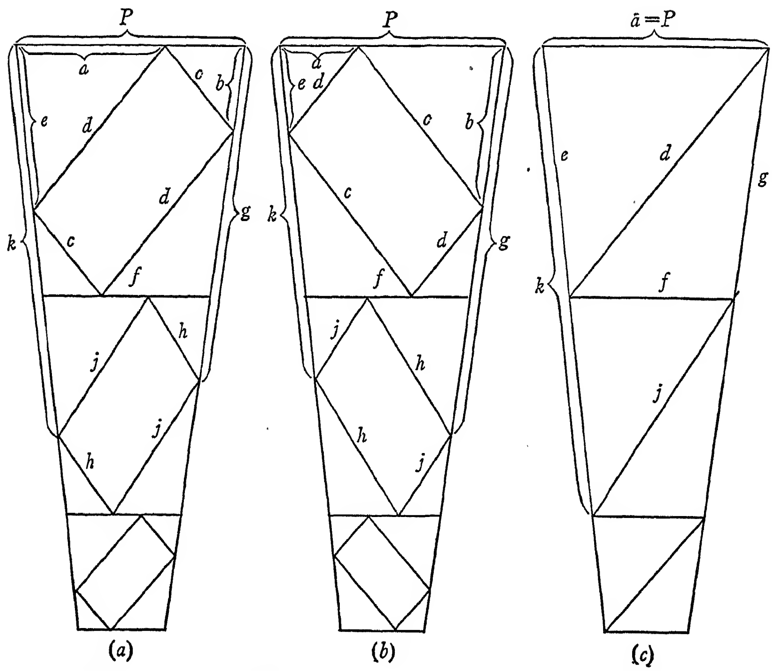


FIG. 68. Various Force Diagrams for Tower Shown in Fig. 66.

determinate, the force carried by the tension diagonal is proportional to *d*, and the one carried by the cross strut is proportional to *f* (Fig. 68*c*). This is the 100–0 rule of cross-bracing design. If the cross strut *CD*, Fig. 67*a*, is omitted, the truss is also statically determinate, the force *f* in Fig. 68*a* or 68*b* equals zero, and the force parallelogram *dcdc* has equal sides and thus assumes the shape of a rhombus. This exemplifies the 50–50 rule for cross-bracing design.

With the cross strut *CD* (Fig. 67*a*) in place and the compression diagonal given credit for what it is worth, then the truss is redun-

dant, and the theory of static equilibrium requires supplementing before a complete solution can be made.

In order to proceed on the basis of the elastic-stress theory of strength (item 1, p. 118) we make a simplifying assumption that all members of the truss are ideally pin-connected. I make this assumption because it is the kind of assumption which is generally made and is one of which I approve. However, it must not be thought for a minute that this assumption can be justified by elasticity reasoning. It may be argued that the truss should be rigorously analyzed by elasticity reasoning without making any assumptions. Should we attempt this, it will immediately become clear that we are forced to guess about stress raising around bolt holes, degrees of end fixity at the joints, and erection stresses which would inevitably throw any refined elasticity reasoning out of gear. If we thus introduce a limit-design assumption in elasticity reasoning, a common practice, and proceed on the supposition that all members of the truss are ideally pin-connected, then, as we argued on page 89, the force c, carried by the compression diagonal will inevitably be greater than the force d, carried by the tension diagonal (Fig. 68b). In order to arrive at a quantitative solution, we must begin by assuming the size of all members of the truss. We then modify these members in the light of our analysis and by successive approximations arrive at a satisfactory solution.

In most cases, if the compression diagonal is designed to support its share of the load safely, if the truss is assumed to function elastically, and the tension diagonal is made the same size as the compression one, then the tension diagonal will be very uneconomically understressed. The 100–0 rule of design—the rule based on the assumption that it is more economical to ignore the compression diagonal altogether than to give it credit for its contributing share to the strength of the truss, and that, if we so ignore it, we surely err, but we err comfortably on the safe side—is a kind of rough-and-ready limit design, and is a recognition of the unsatisfactory results obtained by the best possible elastic analysis.

The other alternative strength theory, the theory of limit design (item 3, page 118), capable of supplementing the laws of equilibrium so as to arrive at a fairly complete picture of strength, ignores completely any stresses and deformations that may exist within the elastic range. We have called attention to the fact that the three diagrams in Fig. 68 are derived from static-equi-

librium reasoning. They are qualitatively correct but quantitatively indeterminate. This statement may be compared with what we said, on page 29, about the bending-moment diagram for a built-in beam subject to uniformly distributed load. We argued in that case, again on the basis of equilibrium reasoning, that the bending moment for the built-in beam is identical to that for the simple beam except for the constant of integration. This constant of integration is represented by the position where the X axis is placed. The X axis may conceivably be placed in any one of an infinite number of positions. Elasticity reasoning places it so as to make the moment at the ends of the built-in beam twice as large as the moment at its midpoint, whereas limit-design reasoning argues that the moment at the ends and the one at the midpoint are equal. In either case the moment at the end of a built-in beam subject to uniformly distributed load and the moment at the middle add up to $wl^2/8$. It is of interest to note that the sides c and d in the parallelogram of either Fig. 68*a* or Fig. 68*b* add up to equal the diagonal marked d in Fig. 68*c*. This means that the two diagonals together are loaded with a definite portion of the load P, regardless of how this load is distributed between them. If the load P on the tower is gradually increased, the force diagram within the elastic range will qualitatively appear similar to Fig. 68*b*. As the buckling strength, say of diagonal c, is reached, its resistance is frozen. Subsequent increases in the load P will result in a qualitative change in the force diagram from Fig. 68*b* to Fig. 68*a*. Note that quantitatively c in Fig. 68*a* is substantially equal to c in Fig. 68*b* and that P in Fig. 68*a* is considerably larger than P in Fig. 68*b*.

Discussion of Assumptions. The crux in the foregoing discussion lies, of course, in the assumption that, once the buckling strength of the compression diagonal is reached, its carrying power remains constant until some other member, say the tension diagonal or the cross strut CD, is also loaded to its capacity strength. It seems to me that this assumption is justified for the following reasons:

1. The Canadian Bridge Company, to the best of my belief, the foremost firm designing, constructing, and erecting transmission towers, has for 40 years designed them by means of limit design, sold them in nearly every country in the world, and by hundreds of tests of actual towers established a satisfactory relationship between their design procedure and test results.

2. Columns in the long-column range, see curve for $l/i = 306$ in Fig. 51, behave, with a very high degree of precision, like curve ABC in Fig. 1, with this added advantage that the behavior *is perfectly elastic.* In tests of towers it is frequently observed that compression members take on a pronounced wow as the loads continue to increase but apparently straighten out perfectly when the loads are decreased. The dead weight becomes a factor if columns are very long and are placed in a horizontal position.

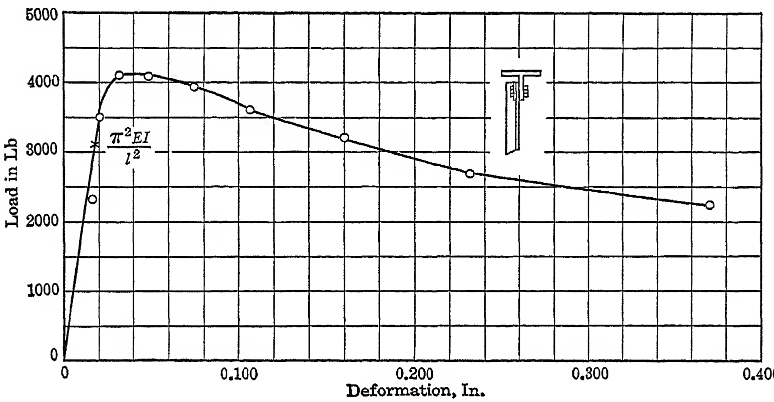


FIG. 69. Compression Text of $1\frac{3}{4}'' \times 1\frac{3}{4}'' \times \frac{1}{8}''$ Structural-Steel Angle—Simulating End Fixity Conditions of Cross Bracing in a Transmission Tower. l/i for specimen was 200.

The inclusion of this dead weight in the computation of the limit strength of a column presents no difficulty (see reference 1*g*). In general, it may be stated that the limit column strength of columns in the long-column range can be computed with a very high degree of accuracy, and this limit column strength remains constant under a comparatively large degree of deformation.

Figure 69 represents a load-axial deformation record of a $1\frac{3}{4} \times 1\frac{3}{4} \times \frac{1}{8}$-in. angle ($l/i = 200$) tested as a column, and loaded through $\frac{5}{8}$-in. bolts in $\frac{11}{16}$-in. holes. This represents the type of connection detail common in transmission towers. The maximum load is 32 per cent in excess of the Euler load, $P = \pi^2EI/l^2$, due, no doubt, to the end fixity supplied by the connection. It is worthy of note that when the axial deformation is tenfold that of the elastic deformation, the load supported by the column is still $P = \pi^2EI/l^2$. Thus the load-deformation curve of Fig. 69, simulating service conditions, is not materially

different in its essential aspects from the ideal pin-ended-column curve for $l/i = 198$, shown in Fig. 51.

The dubious element centers around columns in the intermediate column range and particularly around columns of critical column length l_1. The only experimental evidence I have available and know to be available is the curve for $l/i = 100$ in Fig. 51. In connection with the development of formula IX it was argued that we cannot expect to obtain an elastic-instability condition anywhere except in a laboratory. In any practical case, as the result of inevitable eccentricities, the ductile-equilibrium condition is approached gradually and not suddenly. Neither theory nor test results show a constant load-carrying capacity after ductile equilibrium has been established in case of columns of critical length l_1. However, tests indicate (Fig. 51), for columns in the intermediate-column range, a rather gradual decrease in load-carrying capacity of columns as deformations are increased. The column behavior, after buckling, of columns in the intermediate-column range, is not what we may ideally wish it to be. Neither is it so bad as to invalidate limit-design philosophy completely. Formula IX automatically provides a variable factor of safety which increases as the critical column length l_1 is approached from either side.

As to the convenient "100–0 rule," it may be justified, if at all, on grounds that it results in a large error, comfortably "on the side of safety." It cannot be justified on grounds that it avoids compression members in the cross bracing. The compression in the cross strut CD, represented by f on all three diagrams in Fig. 68, is largest in the case when compression diagonal C is considered as inoperative (Fig. 68c). It is a tensile force in Fig. 68b which represents elastic-force distribution and is a compression force in Fig. 68a which represents limit-force distribution. From Fig. 68 it may be seen that if $c = d$ and $h = j$ then the central parallelograms are rhombuses. They meet at a point and force $f = 0$. This then represents a force diagram for a truss without the cross brace C–D. With the cross strut C–D in place, it may be seen from Fig. 68a that the more load carried in compression by member c, the less load needs to be carried by member f. In any case, either member c or f must carry compression. It strikes me as advisable to give both members c and f credit for what they are worth.

Design of Towers. The cross bracing of transmission towers is for the purpose of reducing the effective length of the legs as col-

umns. As the direction of the wind as well as the determination of which cable may break, if any, is unpredictable, all members in a tower may be stressed in compression and thus may act as columns. The diagonal bracings are connected at the point where they cross. The brace in tension effectively restrains the brace in compression at this point.

If a represents the shorter and b the longer of the two lengths into which the compression diagonal is divided, then formula X represents what corresponds to the Euler buckling load of the compression diagonal (see page 116). Formula X is approximate but the error is slight and is on the safe side. For small values of a/b, say $a/b < 0.1$, the problem is physically unrealizable, and thus all attempts at rational analysis, including formula X, become futile. If formula IX is used then P/A of formula X may be substituted for s_{cr} in formula IX.

If the safe load for which the truss is to be designed is $P/2$, and the factor of safety equals 2, or the overload factor equals 100 per cent, then the limit load equals P. This limit load P is laid off to scale as shown in Fig. 68*c*, and two lines are drawn parallel to the legs as shown. If next a line d is drawn, as shown in Fig. 68*c*, parallel to the diagonal d in Fig. 67*a*, then the length of the line d in Fig. 68*c* represents, to scale, the sum of the forces to be carried by the tension and compression diagonal. Next we select two diagonals; the limit column strength of one plus the limit tensile strength of the other equals the force represented by d in Fig. 68*c*. Next we draw the parallelogram as shown in Fig. 68*a*, where the single force vector d is replaced by a parallelogram composed of the tensile force vector d and the compressive force vector c. For member a in Fig. 67*a*, a strut is selected the limit compressive strength of which is equal to or slightly greater than the force vector a in Fig. 68*a*. To select a strut of a strength considerably in excess of the force vector a, in Fig. 68*a*, would serve no useful purpose, as in that case the diagonals would surely fail long before strut a was endangered. What applies to member a applies with equal force to all other members. What applies to the top panels applies equally to all other panels. Diagonals are selected so that their combined tensile and compressive strengths are sufficient to develop the force necessary for diagonal bracing. If we had an infinite choice of structural members from which to make our selections, we might come close to a design of the type of "the one hoss shay." Limited as we are in our choice of mem-

bers we will most likely overdesign; that is, our completed design will represent a structure capable of resisting a force somewhat in excess of the force P.

Professor Kist suggested this method of design as formulated on page 39, although to my knowledge he had no intention of applying it to compression members.

Chapter 5

CONNECTION DETAILS

We have seen how ductility is of primary importance in the strength determination of redundant structures. Ductility of material, however, is not enough. Structural members, as individual units, should behave, for the philosophy of limit design to be applicable, according to the load-deformation pattern as illustrated by Fig. 1. The behavior of a structural member, as a unit, is profoundly affected by the manner in which it is connected to other structural members. The rule that connections should be detailed so as to develop, as nearly as possible, the full strength of the member is sound and of long standing. The parallel rule, that thought should be given to developing, as nearly as possible, the full ductility of the member is not so well established.

The cross bracing of transmission towers generally consists of equal-legged angles of mild steel. These angles are bolted to the legs of the towers through $\frac{5}{8}$-in. bolts in $\frac{11}{16}$-in. holes. The holes are punched through one of the legs of the angle. This, to my view, is an example of an extremely bad connection detail. The fact that it is being used can be accounted for only because the material of which towers are constructed is excessively ductile. I venture to predict that, if a similar connection of aluminum angles, for example, were attempted, it would prove to be highly unsatisfactory.

A tensile force applied to an angle by means of a bolt passing through a hole in one of the legs represents two very bad features: (1) The load is applied with a very great eccentricity, and (2) an excessive stress concentration results near the hole and close to the edge of the angle leg. The ductile flow of which the material is capable will thus be concentrated around the bolt hole. In fact, the angle will rupture through the hole before the elastic-limit stress in the main portion of the member is reached. Therefore, in spite of the great ductility possessed by the material the member

as such will behave as a brittle member. This point is illustrated by Figs. 70 and 71.

The tests represented by Figs. 70, 71, and 72 and by Table 1 *

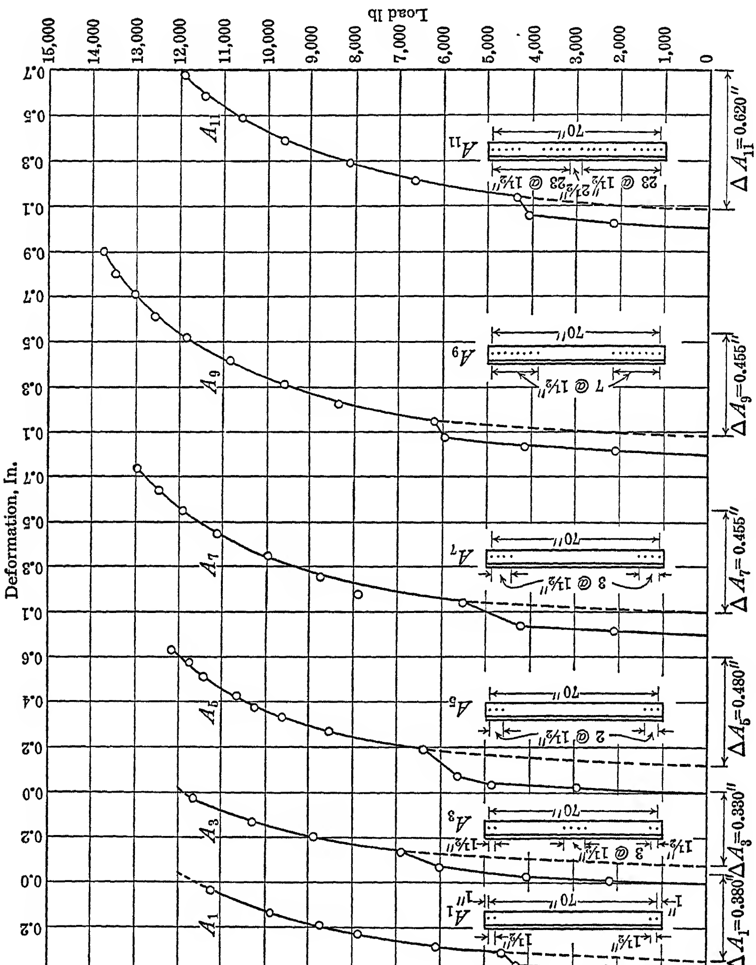


FIG. 70.

were made on 1¾ x 1¾ x ⅛-in. steel angles. The loads were applied through ⅝-in. bolts fitted into 11/16-in. holes, similar to conventional connections in transmission towers. In the tension tests no attempt was made to take up the 1/16-in. slack between

* Figures 69, 70, 71, and 72 and also Table 1 are reproduced from reference 1*d*.

bolt and hole. The abrupt breaks in the load-deformation curves, Figs. 69 and 71 represent the loads at which friction was overcome and slip took place. In the compression tests, the play in the

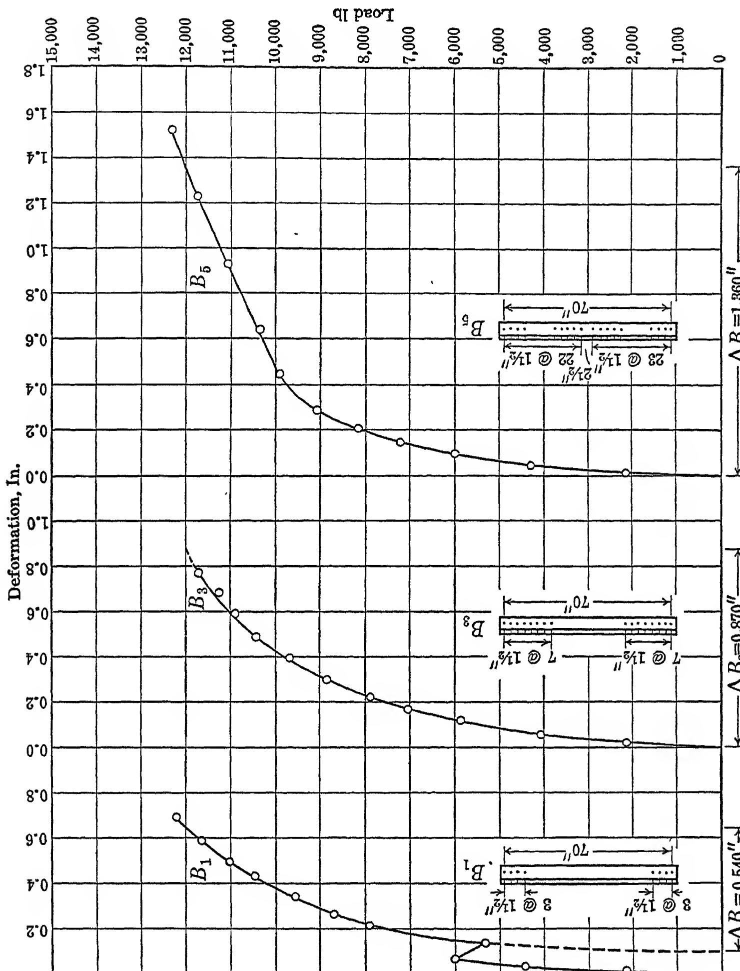


Fig. 71.

holes was taken up before the tests were begun. The lengths of both compression and tension specimens, center to center of bolts through which the load was ransmitted, was 70 in.

The behavior of the conventional angle connection, in tension, is represented by A_1, Fig. 70. By adding one extra hole in the same leg at each end, for the purpose of distributing the ductile

flow, the ultimate strength was not affected but the deformation of the member was increased from 0.380 to 0.480 in. or 26 per cent. If the entire angle leg is perforated with extra holes, curve A_{11}, Fig. 70, then the ductility is increased from 0.380 to 0.620 in. or

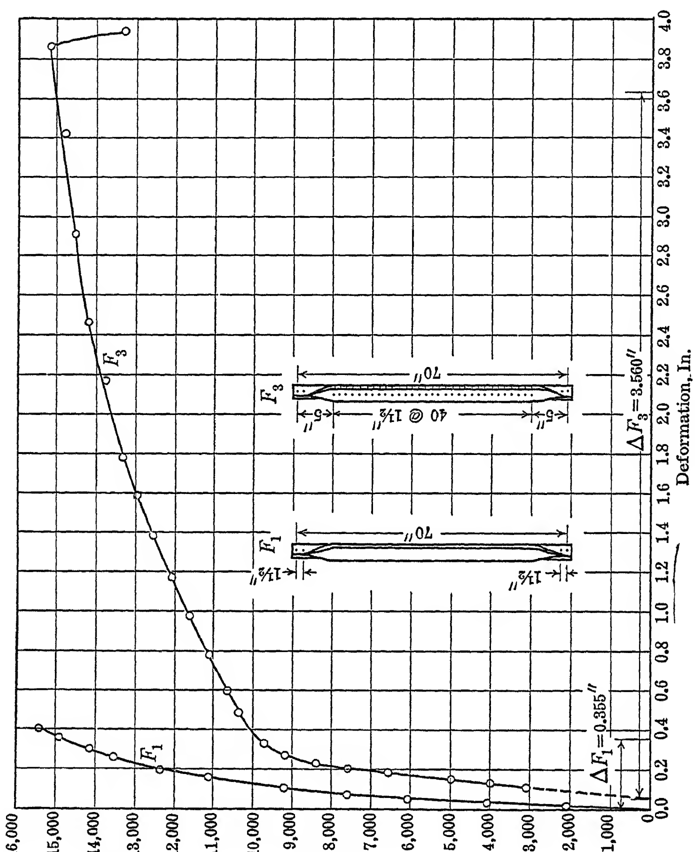


FIG. 72.

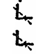
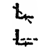
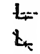
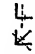
63 per cent. If both legs of the angle are perforated with holes throughout their entire length, B_5, Fig. 71, then again, the ultimate strength is not affected but the ductility is increased from 0.380 to 1.360 in. or 260 per cent.

We have made a plea that full advantage be taken of the compressive strength of cross bracing. It remains thus to be determined how the extra perforation of angle legs, introduced for the purpose of increasing the ductility of tension members, affects

the strength of these same members in compression. In reference 1*d*, the odd numbers represent tension tests, and the even numbers compression tests. Thus A_{12} in Table 1 represents an angle iron, with one leg perforated with extra holes, identical to the one represented by the sketch for A_{11} in Fig. 70. The 43 perforations in a single leg of the angle A_{12}, Table 1, result in a decrease in compressive strength from 4080 to 3660 or 11 per cent. However,

TABLE 1. COMPRESSION TESTS OF 1¾ BY 1¾ BY ⅛-IN. ANGLES

All Angles Were Loaded Through One ⅝-In. Bolt at Each End. Spaced 70 In. Center to Center. All Holes Were ¹¹⁄₁₆ In.

Specimen	*Number of Extra Holes*	*Capacity Load in Lb*	*Buckled About Axis Marked*	*Load Corresponding to Deformation of 0.400 In.*
A_2	0	4,080		2,180
A_4	4 *	3,970		2,080
A_6	2 *	4,100		2,150
A_8	4 *	4,330		2,300
A_{10}	12 *	4,110		2,280
A_{12}	43 *	3,660		2,140
B_2	8 †	4,230		2,300
B_4	24 †	3,880		2,450
B_6	86 †	4,850		2,450
F_2	0	5,150		1,820
F_4	82 †	5,280		1,870

* Extra holes in one leg only.
† Extra holes in both legs.

the 86 perforations, distributed through both legs of the angle B_6, Table 1 (identical with B_5, Fig. 71), result in an increase in compressive strength from 4080 to 4850 or 19 per cent.

Angles with both ends crimped and perforated represented by F_3 in Fig. 72, may not be usable as cross bracing because of difficulty in effecting connections at the point of crossing. However they may effectively be used as single bracing. As such, they are superior to all other angles with different types of end connections. The ends may be shaped at low cost. The two legs of the angle may simply be pressed together cold, if provision is made for keeping the ends in alignment with the axis of the member. Specimen F_3 (Fig. 72) shows an increase in tensile strength over specimen A_1 (Fig. 70) from 11,800 to 15,200 or 29 per cent; an increase in compressive strength from 4080 to 5280 or 29 per cent; an increase in

ductility from 0.38 to 3.8 or 900 per cent; and a decrease in weight per unit length of 15 per cent.

It is confidently predicted that $1\frac{3}{4} \times 1\frac{3}{4} \times \frac{1}{8}$-in. equal-legged aluminum angles, when perforated with extra holes, will show superior ductile properties over the conventional unperforated steel angles.

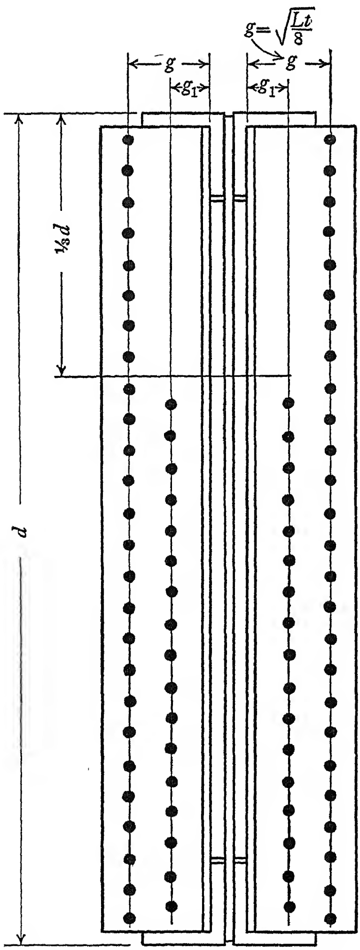


FIG. 73. Stringer-Floor Beam Connection.

Another example of ductile deficiency in a connection detail is the conventional connection of a stringer to a floor beam in a bridge. The connection is designed for shear, not for moment. At a point where normally the moment would be very large, it is assumed to be zero. According to Kist's dictum (expressed on page 39) this is permissible. However, there are always exceptions to rules. Kist predicated his dictum on the assumption that a measure of ductility can be counted on. Structural steel, the material of which railway bridges are constructed, is extremely ductile. The conventional connection between stringer and floor beam is not. To my knowledge these connections have consistently given trouble for the last 50 years. The reason they have persisted throughout these years is no doubt due to the large measure of ductility in the material. I have no doubt that if similar connections were used in aluminum construction they would not merely work loose and require replacement but would be also entirely impractical. In 1940 Mr. W. Deubelbeis, assistant engineer of the Canadian Pacific Railway, used the device of

leaving out the top rivets in the inner row of connecting angles (Fig. 73).* In the same year Professor W. M. Wilson recommended this procedure, fortified by test records, in a paper published in the Proceedings of the American Railway Engineers Association (reference 11).

Ductility is generally considered as a highly desirable property in shops where structural members are manufactured. We believe it is more than that. It is a property essential to the satisfactory functioning of completed structures. It is important that such measure of ductility as a structural material may possess be not unnecessarily taxed. The closest scrutiny of the design of connection details, with the view that the ductility of the connected member may be preserved commensurate with the ductility of the material, is essential. Certain standard steel connection details may well be reconsidered. They inevitably will have to be reconsidered when an attempt is made to apply them to structures built of materials less ductile than steel.

* Figure 73 is reproduced from reference 11.

Chapter 6

EVALUATION OF LIMIT DESIGN

On the occasion when reference 1*k* was presented in New York City, Professor Hardy Cross expressed himself to the effect that either he knew nothing about limit design or else he had known it all his life. I am inclined to believe he has known it all his life. Whether he has faith in it as some of us do is another matter. Certainly, his end-moment-distribution method is a beautiful simplification of elasticity reasoning, but is elasticity and stress-analysis reasoning nevertheless. The end-moment-distribution method is a denial of the limit-design philosophy set forth in this book.

The pioneers in strength theory, da Vinci and Galilei, directed their attention to limit strength. Even Euler, in his phenomenal treatise (reference 10), made no allusion to stresses. Many of us differ in our sense of value. Mathematical analysis seems frequently to be regarded as the essence of a theory. Assumptions, with equal frequency, are advanced with little or no discussion or logical justification. I definitely regard the assumptions underlying any theory as the most basic and important elements of a theory. When we say $s = P/A$, the following assumptions are involved:

1. The load is concentrically applied.
2. The bar is free from residual stresses.
3. The bar is free from scratches.
4. The material is perfectly homogeneous.

Of these four assumptions the fourth one is the only reasonable one. The first three represent wishful thinking and, as assumptions in an elasticity philosophy, are not justified. As a limit-design formula, the only assumption involved in establishing $s = P/A$ is the one that presupposes that, before deformations of disastrous proportions can take place, stresses will be uniformly distributed over the cross-section area and will not be materially

affected by minor eccentricities or by the presence of residual stresses or scratches. Our most elementary strength formula $s = P/A$, thus, is justified by limit design and not by elasticity reasoning.

So-called secondary stresses are commonly ignored. If taken seriously, they would condemn numerous sound engineering practices. In ignoring them, we refute elastic-stress-analysis reasoning. When, however, we resort to area moments, to conjugate beams, to slope defection, to end-moment distribution, to elastic-energy theory, or if we turn to a handbook and accept the conclusion that a beam over three supports is no better than two simple beams of equal span length, then we deny limit-design reasoning.

Limit-design philosophy is supported more by numerous rules and sound engineering practices than by the few experimental test records presented in this book. I would like to see these rules logically justified, rather than dogmatically presented. Then such logical justification may be extended beyond the limited fields to which the rules apply.

Perforated and crimped angles, as structural members, for example, may be economically justifiable or not. In structural-steel practices we have been blessed with such a very large amount of ductility that connection and construction details have been developed which cannot be applied to either aluminum or magnesium construction. Should we attempt to build transmission towers with equal-legged angles of aluminum or magnesium alloys, we would either have to improve the ductility of the structural members of the tower (see Fig. 72) or have to give up the attempt.

The theory of limit design has its limitations. Should we undertake the building of structures with brittle materials such as glass, then limit-design philosophy would not be applicable. Along with limit design we would then have to scrap numerous simplifying rules and formulas, such as: that stresses may be assumed to be equally distributed over rivets; that riveted or welded trusses may be analyzed as pin-connected trusses; that secondary stresses, residual stresses, and erection stresses may be ignored; that $s = P/A$, and numerous other formulas. In cases in which limit stresses may be repeated many thousands of times, or reversed hundreds of times, limit-design principles may not be applicable.

In this book only the simpler and the more important cases of the strength phenomena have been discussed, namely, direct tension, direct compression, and bending. Generally, these phe-

nomena, to which should be added the shear phenomenon, occur simultaneously in varying degree. A wide field still remains open for further theoretical analysis. Further research and further experimental evidence are needed in connection with strength problems which involve combined loadings such as the strength of thin web structures (see page 59), and unsymmetrical bending.

Personally, I regard the overemphasis on elastic stress analysis of the last 50 years as an aberration which, I believe, shows signs of having run its course. I believe it unsound as an ultimate criterion of strength for most redundant structures. Limit design, as a reversion to type, it seems to me, holds promise.

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Appendix

PROBLEMS

In the following problems assume the stress–strain relationship to be as represented by Fig. 1.

Assume the tensile and compressive elastic-limit stress to be $s_1 = 36{,}000$ psi.

Assume the shear elastic-limit stress to be $s_{s_1} = 12{,}000$ psi.

Assume the modulus of elasticity to be $E = 30{,}000{,}000$ psi.

Assume the modulus of rigidity to be $G = 12{,}000{,}000$ psi.

1. A steel beam of 2 x 6-in. cross-section area and 72 in. long is built in at both ends and loaded with a uniformly distributed load w lb per ft.

(*a*) What is the load w_1 that can be supported by the beam when the elastic limit stress s_1 has just been reached?

(*b*) What is the deflection of the midsection of the beam under a load w_1?

(*c*) What is the load w_2 that can be carried by the beam when the elastic-limit stress s_1 is just reached in the middle of the beam? (Assume the full ductile resisting moment to be developed at the built-in ends.)

(*d*) What is the deflection of the midsection of the beam under the load w_2?

(*e*) What is the residual bending moment when the load w_2 is removed?

(*f*) What is the residual deflection of the beam's midsection when the load w_2 is removed?

(*g*) What is the limit load w_3 which will induce total collapse of the beam?

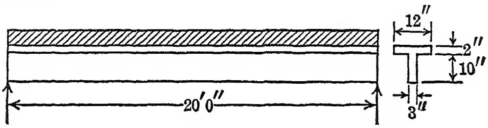
(*h*) What is the maximum uniformly distributed load w_4 that a 2″ x 6″ x 6′ 0″ simply supported beam can carry when the elastic-limit stress s_1 is just reached?

(*i*) What is the deflection of the simple beam of problem 1*h* under a load w_4?

Answers

(*a*) $w_1 = 12{,}000$ lb per ft	(*d*) $\Delta_2 = 0.151$ in.	(*g*) $w_3 = 24{,}000$ lb per ft
(*b*) $\Delta_1 = 0.0648$ in.	(*e*) $M = 6000$ ft-lb	(*h*) $w_4 = 8000$ lb per ft
(*c*) $w_2 = 20{,}000$ lb per ft	(*f*) $\Delta = 0.043$ in.	(*i*) $\Delta_4 = 0.216$

2. A simply supported T beam (dimensions as shown) is loaded with a uniformly distributed load w lb per ft.



(*a*) Find w_1 the load which will induce in the beam the elastic-limit stress s_1.
(*b*) Find the limit load w_2.
(*c*) Find the maximum deflection under condition (*a*).

Answers

(*a*) $w_1 = 5775$ lb per ft (*b*) $w_2 = 10{,}300$ lb per ft (*c*) $\Delta = 0.941$ in.

3. A T beam of the same cross-section dimensions as the one shown in problem 2, is built in at one end, freely supported at the other end, and loaded with a uniformly distributed load w lb per ft.
(*a*) Find w_1, the load which will induce in the beam the elastic-limit stress s_1.
(*b*) Find w_2, the load which will induce in the beam, between supports, the elastic-limit stress s_1. (Assume the full ductile resisting moment to be developed at the built-in end.)

Answers

(*a*) $w_1 = 5775$ lb per ft (*b*) $w_2 = 10{,}200$ lb per ft

4. A solid round shaft of mild steel is twisted beyond the elastic stress range. Write an expression for the angular deformation as a function of the torque T. Let R represent radius of shaft and r the distance from center of shaft to the boundary between elastic and ductile portions of shaft. Plot a s–T graph for a 6 in. solid round shaft of a length 10′ 0″

Answers

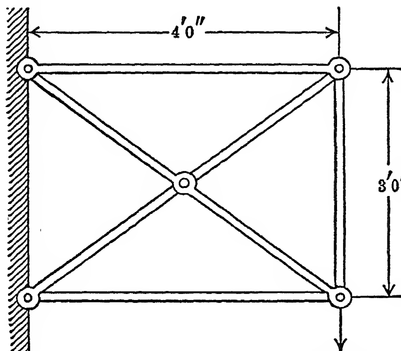
$$T = \frac{s\pi}{6}\,(4R^3 - p^3)$$

$$\theta = \frac{sl}{Gr}$$

5. The truss consists of five square eye-bars, all of same size and of cross section a x a. The cross bars are continuous but are pinned where

they cross. In computing the weight of the truss (weight of steel is 0.283 lb per cu in.), ignore the extra weight in the eyes and assume length of bars as equal to that of center to center of pins.

(*a*) Assume the compression diagonal to be ineffective. Compute the weight W_1 of the truss.



(*b*) Assume all bars to be effective. Compute the weight W_2 by means of theory of elasticity on assumption that elastic-limit stress is just reached in one of the bars. Compute column strength by means of formula, $P = \pi^2EI/l^2 = \pi^2Ea^4/12l^2$.

(*c*) Assume all bars to be effective. Compute the weight W_3 of the truss by theory of limit design. Compute column strength by means of formula.

Answers

(*a*) $W_1 = 113$ lb (*b*) $W_2 = 82$ lb (*c*) $W_3 = 65$ lb

6. Assume all members in the truss shown in problem 5 to be 1-in. round steel bars, pin-connected at the joints. By means of formula IX compute the limit load acting in the place of the 20,000-lb force.

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